Analysis of strip plate on elastic foundation using a generalized subgrade model

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ABSTRACT
A generalized subgrade model is presented by Worku (first author) for subgrades idealized an elastic layer overlying a rigid base. By implementing this model a governing differential equation of strip plate on elastic foundation is formulated. A closed form particular solution of infinite and finite strip plate, when using winkler type and Kerr equivalent Pasternak model are obtain by using different boundary conditions under different loading conditions. A finite element based Plaxis 2D software is used to calibrate the subgrade models. A numerical illustration is provided using this model in comparison with Plaxis 2D models. The result show that the calibrated variants give a good results in agreements with finite element outputs .consequently, the calibrated models can be used in routine analysis of strip plates on elastic foundation and can be incorporated in commercial software.

RÉSUMÉ
Un modèle de sous-sol généralise est présente par Worku (premier auteur) pour les sous-niveaux idéalises d'une couche élastique recouvrant une base rigide. En mettant en œuvre ce modèle, une équation différentielle régissant la plaque de bande sur la fondation élastique est formulée. Une solution particulière de forme fermée de plaque a bande infinie et finie, lors de l'utilisation du type Winkler et du modèle Pasternak équivalent Kerr, est obtenue en utilisant différentes conditions aux limites dans différents conditions de chargement. Un logiciel de Plaxis 2D a éléments finis est utilisé pour calibrer les modelés de fondation. Une illustration numérique est fournie en utilisant ce modèle en comparaison avec les modèles 2D de Plaxis. Le résultat montre que les variantes calibrées donnent de bons résultats dans les accords avec les sorties d'éléments finis. Par conséquent, les modèles calibrés peuvent être utilisés dans l'analyse de routine des plaques de bande sur fondation élastique et peuvent être incorporés dans un logiciel commercial.

1 INTRODUCTION

Plates on elastic foundation are regularly used in civil or mechanical engineering works, such as building infrastructures, roads, railroad, storage tanks or silos foundations, aerospace engineering etc. The key issue in the analysis is modeling the contact between the structural elements- the plate or beam-and the soil bed (soil-structure interaction (SSI) problem).

There are two approaches to develop analytical subgrade models, namely continuum and mechanical. Elastic continuum models typically idealize the subgrade as an elastic medium and specifically as a layer overlying a rigid base, and involve three parameters consisting of the elastic modulus, the Poisson's ratio and the layer thickness. It is apparent that all continuum models available make certain simplifying assumptions to ease the mathematical work involved in the process of devising the models. These models have the advantage that the elastic constants can be established from tests but suffer from a common shortcoming that they are difficult to apply directly. While in mechanical models, the higher order models were devised with the intention of improving on the drawbacks of the simplest and long- enduring Winkler single-spring-bed model by introducing additional elements to ensure shear interaction among the springs that is missing in Winkler's model. However, mechanical models suffer in general from a major common drawback of not suggesting ways of estimating the model parameters.

Recently, a newly developed generalized continuum subgrade model has been introduced by Worku (2010). It considered all stress, strain and displacement components unlike other models proposed in the past. Based on this model, this paper tries to give analytical solutions for strip plates under basic loading cases. Furthermore, it also tries to calibrate the model parameters and to compare the results with the classical models and selected FE based modeling software.

1.1 Generalized Model

Worku (2010) proposed a new approach of continuum modeling by addressing the major shortcomings other continuum models. The subgrade is idealized as an elastic continuum of finite thickness H. This approach of the subgrade by an elastic stratum is convenient in developing the model of both an actual elastic stratum of finite thickness overlaying a rigid base as well as for very thick strata commonly idealized as a uniform half space.

All continuum models including the generalized continuum models developed by Worku are sensitive to thickness H of the stratum. When the thickness of the soil layer increases, it may give unrealistic or excessive deformation. In latter case he express H in terms of the foundation width, *B,* as $H = \chi B$ and the coefficient χ determined from comparison of analytical results with results of finite element model. In later section of this paper the calibration factor χ for strip on elastic foundation will be done.

1.2 Synthesis of Winkler Model with the Generalized Winkler Type Continuum Model

By synthesis the mechanical and continuum models Worku (2013) express the Winkler mechanical model parameter in terms of continuum parameters

$$\bar{K}_S = \frac{E_S}{(1-0.4v)H} \qquad [1]$$

Where $\bar{k}_S$ is modules of subgrade reaction or coefficient of subgrade reaction for Winkler, E_S is modules of elasticity of the soil, υ Poisson's ratio of the soil and H layer thickness of the soil.

1.3 Synthesis of Pasternak Model with the Generalized Kerr-Equivalent Pasternak Type Continuum Model

Worku also express the Pasternak model parameters in terms of Kerr equivalent Pasternak continuum model it take the following form

$$\bar{k}_p = \frac{(0.4\upsilon + 0.67)E_S}{H} \quad \& \quad \bar{G}_p = (1.36\upsilon + 2.28)GH \qquad [2]$$

Where $\bar{k}_p$ modulus of subgrade reaction for Pasternak model, $\bar{G}_P$ is coefficient of shear element in Pasternak's model and G is shear modules of the elastic foundation.

2 ANALYSIS OF STRIP PLATE ON ELASTIC FOUNDATION

The differential equation (DE) of a plate of rigidity D on an elastic subgrade when it is subjected to a transverse load q is given by

$$D\nabla^4 w(x) + P(x) = q(x) \qquad [3]$$

In which p is the contact pressure and w is the surface deflection and ∇ is the Laplace operator. However, the governing equation for strip plate will have the form of ODE Equation.

$$D\nabla^4 w(x) + p(x) = q(x) \qquad [4]$$

This is because of a strip plate is one dimensional problem (length along y-axis is very long). Therefore, all the derivative with respect to y are zero and the plate will be only a functions of x.

2.1 Plate on Single Parameter Model

In this case consider bending of a uniformly loaded strip plate subjected to transverse load supported over the entire bottom surface by an elastic foundation.

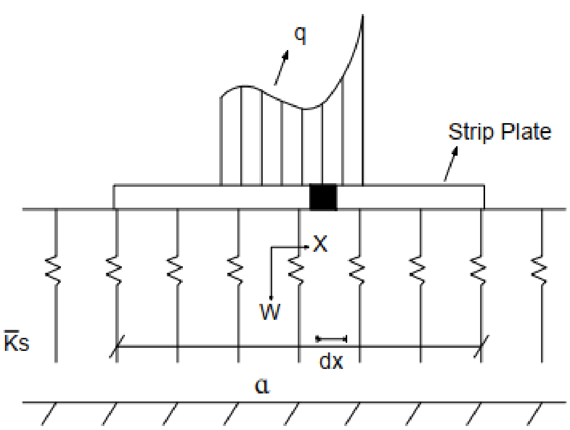

Figure 1. A strip plate on Winkler's mechanical model

$$p = \bar{k}_S w \qquad [5]$$

Now introduce Winkler model and combining equation [4] and [5] the differential equation for the deflection surface of the plate supported on elastic Winkler foundation become

$$D\nabla^4 w(x) + \bar{k}_s w = q(x) \qquad [6]$$

The homogenous equation becomes

$$D\frac{d^4 w}{dx^2} + \bar{k}_s w = 0 \qquad [7]$$

Since the equation [7] is an ODE with coefficient the general solution is obtain as

$$w(x) = \begin{pmatrix} \sin\xi x[C_1\cosh\xi x + C_2\sinh\xi x] + \\ \cos\xi x[C_3\cosh\xi x + C_4\sinh\xi x] \end{pmatrix} \qquad [8]$$

Where ξ is the characteristic width of strip plate $\xi = 2\frac{\beta}{a} = \sqrt[4]{\frac{\bar{k}_s}{4D}}$, $\beta = \frac{\bar{k}_s a^4}{64D}$, a = width of strip plate and C_1, C_2, C_3 & C_4 are open constants.

The open constants can be determine from boundary condition for infinite and finite strip plate when subjected to different loading conditions

2.2 Plate on Two Parameter Model

To improve inherent problems with the Winkler model Pasternak model improve the inherent problem of Winkler model by connecting the ends of the springs by a shear layer, consisting of incompressible, vertical element which can deform only by lateral shear.

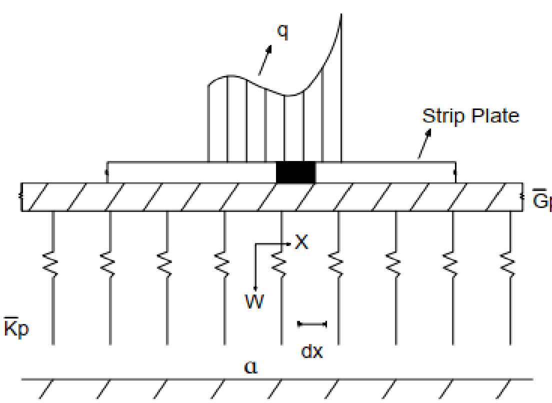

Figure 2. A strip plate on Pasternak mechanical model

$$p(x) = \bar{k}_p w(x) - \bar{G}_p\nabla^2 w(x) \qquad [9]$$

Combining equation [4] and [9] the differential equation for the deflection surface of the plate supported on elastic Pasternak foundation becomes.

$$D\nabla^4 w(x) + \bar{k}_p w(x) - \bar{G}_p\nabla^2 w(x) = q(x) \qquad [10]$$

The homogenous equation becomes

$$D\frac{d^4 w}{dx^4} - \bar{G}_p\frac{d^2 w}{dx^2} + \bar{K}_p w = 0 \qquad [11]$$

To solve this differential equation the following parameters are assumed

$$\beta = \sqrt[4]{\frac{\bar{k}_p a^4}{64D}}, \gamma = \sqrt{\frac{\bar{k}_p a^4}{\bar{G}_p}} \text{ and } \rho = \left(\frac{\beta}{\gamma}\right)^2 = \frac{\bar{G}_p}{8\sqrt{D\bar{k}_p}} \qquad [12]$$

$$\frac{d^4 w}{dx^4} - 64\frac{\beta^2\rho}{a^2}\frac{d^2 w}{dx^2} + 64\frac{\beta^4}{a^4}w = 0 \qquad [13]$$

Since Equation [13] is an ODE with constant coefficients three possible cases of general solution will be obtain depending on whether

$\rho < 1/4, \rho = 1/4$ & $\rho > 1/4$

Case 1 $\left(\rho < 1/4\right)$

$$w(x) = \begin{pmatrix} e^{\xi\phi_1 x}\left(A_1\cos\xi\phi_2 x + A_2\sin\xi\phi_2 x\right) + \\ e^{-\xi\phi_1 x}\left(A_3\cos\xi\phi_2 x + A_4\sin\xi\phi_2 x\right) \end{pmatrix} \qquad [14]$$

Where $\phi_1 = \sqrt{1+4\rho}, \phi_2 = \sqrt{1-4\rho}$ and A_1, A_2, A_3 & A_4 are open constants

Case 2 $\left(\rho = 1/4\right)$

$$w(x) = e^{\xi\phi_1 x}\left(B_1+B_3x\right)+e^{-\xi\phi_1 x}\left(B_2+B_4x\right) \quad [15]$$

Where $\phi_1 = \sqrt{2}$ and B_1, B_2, B_3 & B_4 are open constants

Case 3 (ρ>1/4)

$$w(x) = \begin{pmatrix} e^{\xi\phi_1 x}\left(E_1\cosh\xi\phi_2 x+E_2\sinh\xi\phi_2 x\right)+ \\ e^{-\xi\phi_1 x}\left(E_3\cosh\xi\phi_2 x+E_4\sinh\xi\phi_2 x\right) \end{pmatrix} \quad [16]$$

Where $\phi_1 = \sqrt{1+4\rho}$, $\phi_2 = \sqrt{1-4\rho}$ and E_1, E_2, E_3 & E_4 are open constants.

All the open constants can be determine from boundary condition for infinite and finite strip plate when subjected to different loading conditions.

3 NUMERICAL ANALYSIS AND CALIBRATION

To determine the unknown parameters of mechanical model one can synthesis the two models the same orders will enable to quantify the mechanical model parameters in terms of known parameters of the continuum mode.

Different solution cases were obtained for strip plates on a two parameter subgrade model depending on the value of the parameter ρ. It is important to identify which case represents the most likely scenario for real problems. This can be done by plotting ρ against another carefully selected parameter, incorporating all factors influencing ρ for selected values of thickness of stratum, H.

As it can be seen from equation [12], ρ is a function of the modulus of elasticity of the strip plate (E_p), shear modulus of the soil (G), depth of strip plate (h_p), poission's ratio of the soil (υ) and the thickness of the stratum (H).The effect of these parameters on ρ can be seen by introducing the use of the following dimensionless stiffness factor or relative rigidity of the soil-plate system as suggested by Rajapakse and Selvadurai (1991).

$$k_r = \frac{E_p\left(\frac{h_p}{a}\right)^3}{G} \quad [17]$$

Where Kr =relative rigidity of the soil-plate system. Plots of ρ against Kr are re given in Fig 3 & 4. For clay and granular soils for range of H/a values. The plot revealed that the value of ρ lies in case III ,when the ratio of H/a is greater than 0.85 for clay soils and 0.48 for sand soils .on the other hand, when H/a is less than 0.85 and 0.48 for clay and sand soils respectively, case I should be considered. Due to the fact the ratio H/a less than 0.48 is rare, the most realistic scenario is seen in case III. Therefore, it can be concluded that for all practical purposes it is sufficient to consider only case III.

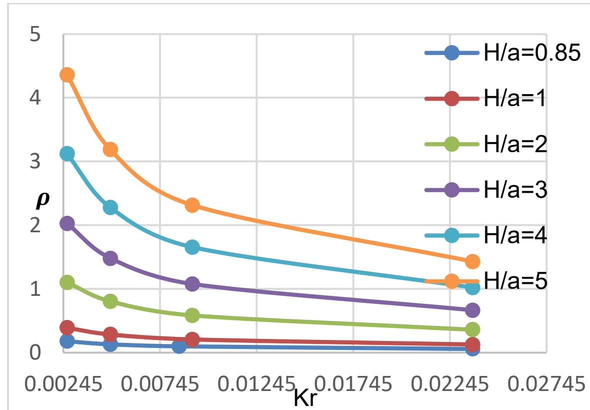

Figure 3. Effects of Kr and H on $\boldsymbol{\rho}$ for Clay soli

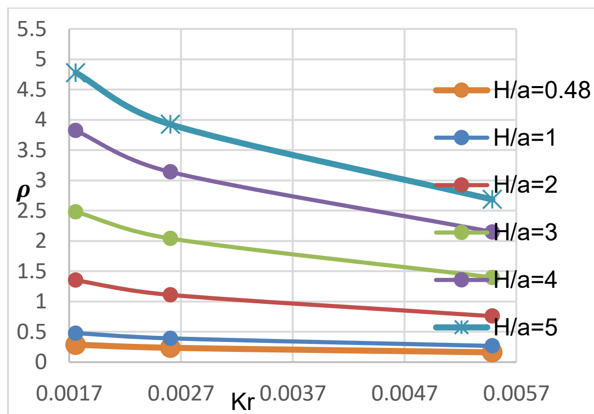

Figure 4. Effects of Kr and H on ρ for Sand and gravel soil

As mention in the above Worku's model are sensitive to the thickness .Therefore, it is important to calibrate this model by using finite element based model. Whereby a calibration factor associated with H is introduced, Plaxis 2D software is used as yardstick.

The first step is finding the optimum mesh size in the for finite element modeling (Plaxis 2D).After fixing the average mesh size the next step is calibration of the Winkler-Type and Kerr-equivalent Pasternak –Type model. This is attained after a number of analysis involving different types of soils, with various relative thickness of the stratum with respect to strip plate width $\chi = H/a$ subsequently, the maximum deflection, w_{max} will be plotted against χ for infinite and finite length strip plate subjected to different loading cases.

In order to find the value of χ, first tangent line is drawn to the Plaxis plot, where it becomes almost constant. Then the intersection point between the tangent line and the corresponding graph of the models becomes the range value of χ. This step is illustrated in Fig 5 and 6. Finally, a calibration factor for each combination of representative type of soil and loading condition is obtained by establishing best fitting trend between the identified range values of χ against characteristic length (ξ).

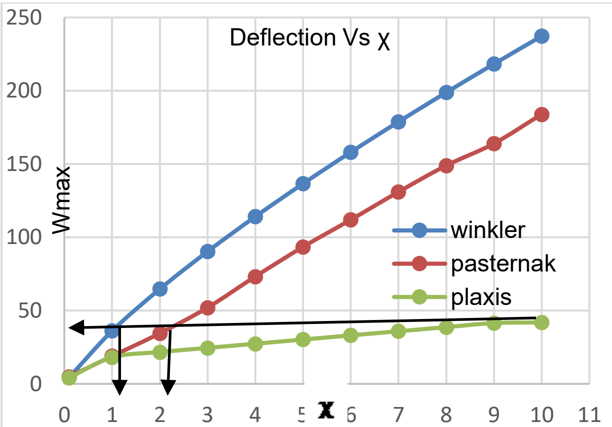

Figure 5. Determination of χ value for long strip plate subjected to selected basic loading types

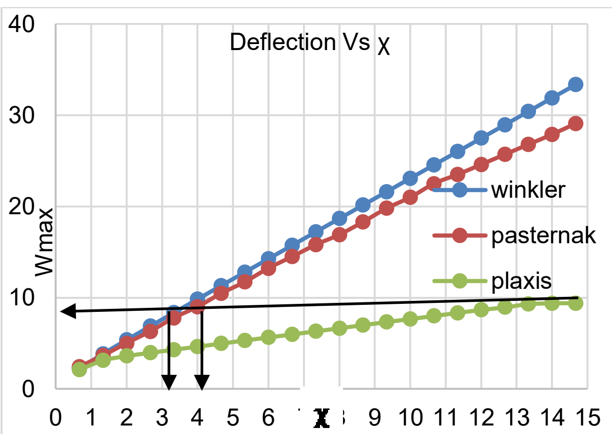

Figure 6. Determination of χ value for shot strip plate subjected to selected basic loading types

Table 1 Summary of calibration factor for Winkler type and Kerr-equivalent Pasternak type model for infinite strip plate

Recommended values of χ for Infinite strip plates on different soil types for the respective loading conditions				
Soil Type	Central Concentrated Load		Uniformly Distributed Load	
	Winkler's Model (χ_w)	Pasternak's Model (χ_p)	Winkler's Model (χ_w)	Pasternak's Model (χ_p)
Soft Clay	1.09	2.45	1.1	2.75
Loose sand	1.04	2.43	1.07	2.73
Medium Stiff Clay	0.95	2.42	0.99	2.66
Medium Dense sand	0.94	2.41	0.98	2.64
Stiff Clay	0.9	2.39	0.92	2.52
Dense Sand	0.89	2.38	0.91	2.51

Table 2 Summary of calibration factor for Winkler type and Kerr-equivalent Pasternak type model for finite strip plate

Recommended values of χ for Infinite strip plates on different soil types for the respective loading conditions				
Soil Type	Central Concentrated Load		Uniformly Distributed Load	
	Winkler's Model (χ_w)	Pasternak's Model (χ_p)	Winkler's Model (χ_w)	Pasternak's Model (χ_p)
Soft Clay	3.83	3.85	3.99	4.12
Loose sand	3.82	3.83	3.95	4.02
Medium Stiff Clay	3.79	3.67	3.82	3.93
Medium Dense sand	3.78	3.65	3.81	3.92
Stiff Clay	3.7	3.52	3.79	3.83
Dense Sand	3.69	3.50	3.78	3.82

The recommended calibration factors for a strip plate resting on different soil types and subjected to selected loading cases as established from best fitting plots are presented in the above tables.

3.1 Calibrated Model Parameters

As stated above, a calibrating parameter is introduced through the relation.

$$H = \chi a \qquad [18]$$

Introducing equation [18] in equation [1] and [2] one can obtain the calibrated model parameters

Winkler type model

$$\bar{k}_S = \frac{E_S}{(1-0.4\upsilon)\chi_W a} \qquad [19]$$

Kerr equivalent Pasternak model

$$\bar{k}_s = \frac{(0.4\upsilon+0.67)E_s}{\chi_p a} \quad \& \quad \bar{G}_p=(1.36\upsilon+2.28)G\chi_p a \qquad [20]$$

3.2 Comparison of Models after Calibration

Numerical comparisons of Winkler type and Kerr equivalent Pasternak model are made with Plaxis outputs for selected loading conditions and for relative rigidities of soil-plate system. The comparison is made for both finite and infinite strip plate. The classification as a long, intermediate and short strip plate is based on Hetenyi suggestion (Hetenyi 1946). Thus, intermediate and short strip plate are treated as finite strip plates.

3.2.1 Infinite Strip plates

Numerical calculation are carried out for infinite strip plates subjected to concentrated and uniformly distributed loads. The comparison deflection curves are presented in for plate and soil properties given in table 3. Representative soft soil and hard soil types are considered. Furthermore a sample example is selected and its Plaxis output for strip plate subjected to point is presented in the following section.

Table 3 Types of soils, strip plate and loading properties considered for the analysis

Types of soils	Elastic modules (E_s)	Poisson's ratio (υ_s)
Loose sand	20000 KN/m^2	0.3
Dense sand	81000 KN/m^2	0.2
Strip plate property		Loading
Elastic modulus , E_p	25Gpa	Concentrated
Width , a	20m	100KN/m
Depth h_p	0.15m	Uniform
Poisson's ratio , υ_p	0.2	50KN/m^2,Loaded region 2m

Table 4 Finite element (plaxis 2D) Inputs

Types of element	Plate model	Soil model
15 nodes	Plain strain model	Linear elastic model

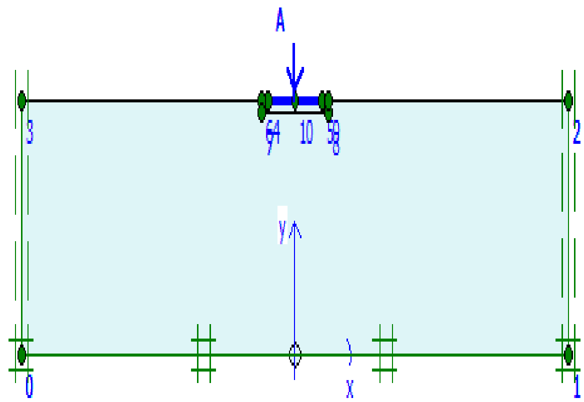

Fig 7. Plaxis Geometry Model

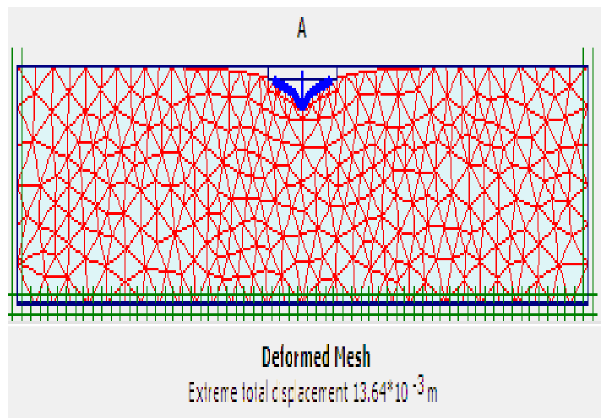

Fig 8.Plaxis model output of deformation mesh for strip plate resting on loose sand soil & subjected to point load

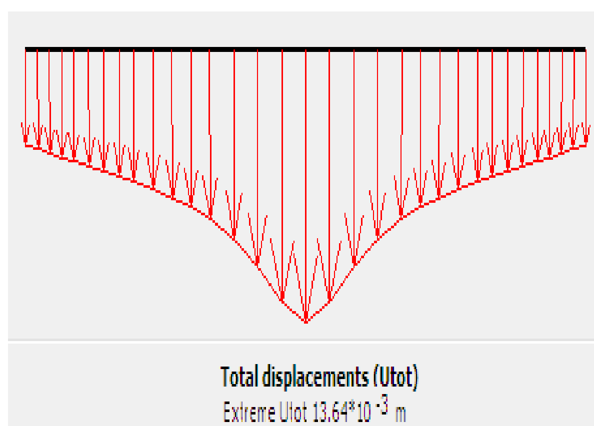

Fig 9.Plaxis model output of Total displacement for strip plate resting on loose sand soil & subjected to point load

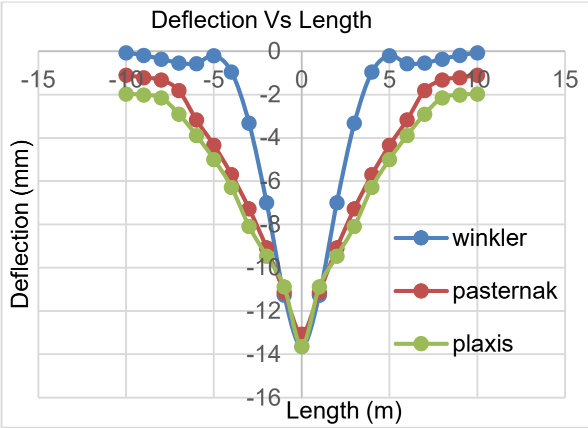

Fig 10. Long strip plate on loose sand soil subjected to a point load

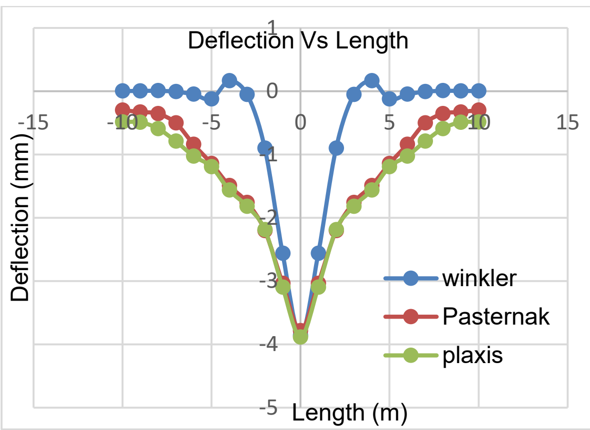

Fig 11. Long strip plate on dense sand soil subjected to a point load

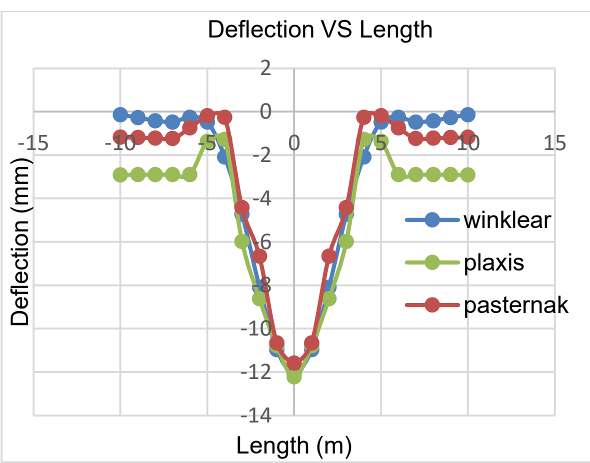

Fig 12. Long strip plate on loose sand soil subjected to a uniformly distributed load

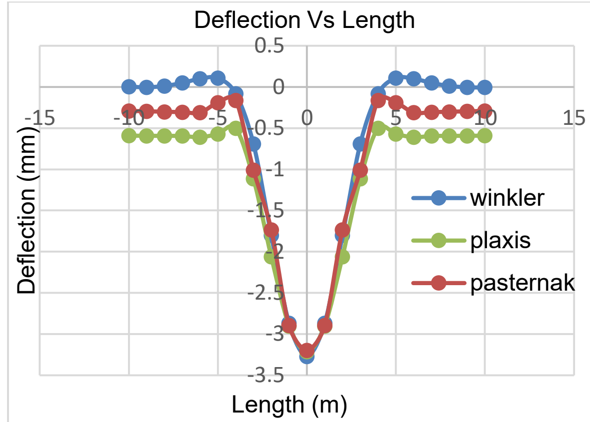

Fig 13. Long strip plate on dense sand soil subjected to a uniformly distributed load

A review of the curves reveals a number of significant observations. The maximum deflection obtained by the two models is in good agreement with the FE based Plaxis model. However, there is a deviation of deflection when moving away from the mid span. This becomes more pronounced in the Winkler-Type model especially in soft soils where deviations may amount to up to 45% in the case of concentrated load and 20 % in uniformly distributed cases. This is due to the fact that Winkler-Type model does not consider the shear interaction behavior of the soil, from the outset despite the calibration. Additionally, because the calibration factor is solely dependent on the maximum deflection, it is certain to have deviations when moving away from the maximum deflection. As far as the Kerr-equivalent Pasternak type model is concerned, a very good agreement with the reference FE based analysis result is obtained over the majority of the plate width.

Though not presented here, a similar trend is observed in the internal moments, the deviation increases when moving away from mid span but ultimately decreases before it reaching the edge of the plate. Whereas in the shear force it is invariably the same for all models.

3.2.2 Finite Strip plates

A similar analysis has been conducted for finite strip plate of 3m length subjected to concentrated load at mid span and uniformly distributed. For the analysis purpose the same properties are used as table 3.

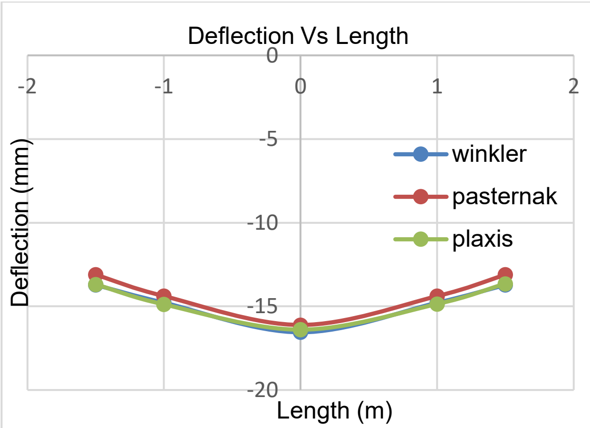

Fig 14. Short strip plate on loose sand soil subjected to point load

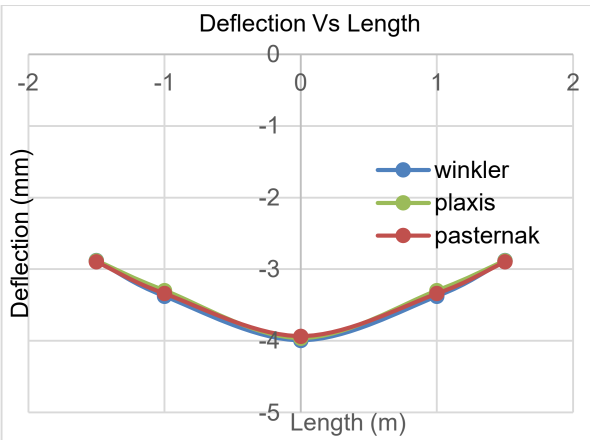

Fig 15. Short strip plate on dense sand soil subjected to point load

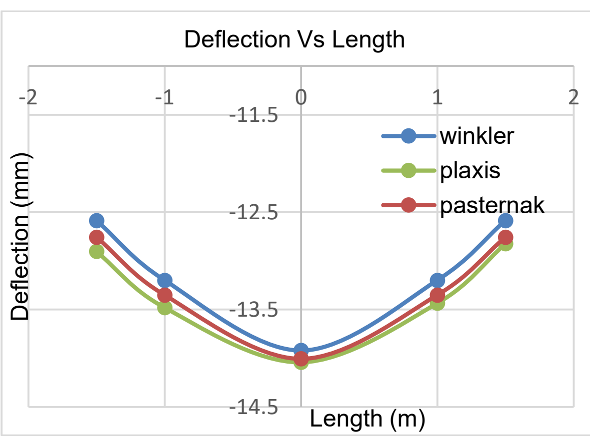

Fig 16. Short strip plate on loose sand soil subjected to uniformly distributed load

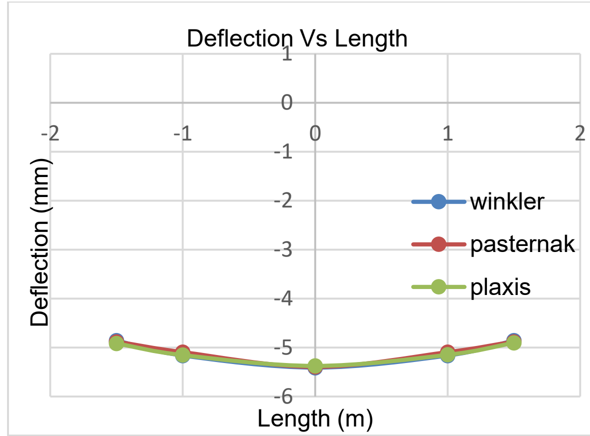

Fig 17. Short strip plate on dense sand soil subjected to uniformly distributed load

As could be observed from the figures, the deflection shows little deviation throughout the plate in all models. For some particular soils, the maximum deflection obtained by Winkler type model gives a slightly higher values in the case of concentrated vertical load. Whereas, the maximum deflection is slightly lower in the case of uniformly distributed load. It should be noted that for Kerr equivalent Pasternak model is in an excellent compliance with Plaxis 2D model outputs.

The differences in the moments, which are not provided here, are smaller and those of the shear are almost negligible.

4 CONCLUTION

A set of generalized closed form solutions for deflection are obtained for strip plate supported by elastic foundation under selected basic load basic load types. The inherent sensitivity of the models to layer thickness has been avoided by calibrating them with respect to the thickness itself normalized with respect to strip plate width. For long strip plate very small deviation in maximum deflection for both models but eventually increases when moving away from mid span. This is more pronounced in Winkler type model, due to the fact the assumption when formulating the model does not include shear interaction behavior of the soil. Moreover, since the calibration factor is linked solely to maximum defection, it is certain to have deviations when moving away from the maximum deflection. In short strip plate deviation of deflection between the models and the Plaxis 2D is very small. The deviation in bending moment modestly increases when moving away from the mid span but ultimately decreases before reaching the edge of long strip plate. In contrary for short strip plates the models are concurrent with slight deviation with FE based software. The shear force is invariably the same for all models. It implies that it is little affected by the type of model used. Generally, in all cases using the recommended calibration factor provided in table 1-2, the Kerr equivalent Pasternak type model gives more compatible results to Plaxis 2D outputs than Winkler type model. Therefore by using this model and the accompanying spreadsheet program, practical analysis of strip plate on an elastic foundation can be easily performed.

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