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Application of a generalized subgrade model in the analysis of circular plates on elastic foundations

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ABSTRACT
A new rigorous and adjustable, yet quite handy, continuum foundation model recently proposed by Worku, the senior author, without neglecting any stress, strain, or deformation component in the subgrade is utilized to analyze circular plates on elastic foundations. The governing differential equations for the case of axisymmetric loading are solved for two variants of the model: Winkler-type and Pasternak-type models. Closed form solutions are presented for different loading conditions on small and large circular plates. The finite-element-based software PLAXIS 2D is used to calibrate the presented model variants by seeking agreement with respect to the maximum deflection. Numerical examples are solved to demonstrate the performance of the models. The outputs show that the proposed models are suitable to conduct analyses of plates and beams efficiently.

RÉSUMÉ
Un nouveau modèle de fondation de continuum rigoureux et ajustable, mais assez pratique, récemment proposé par Worku, l'auteur principal, sans négliger aucune contrainte, déformation ou composante de déformation dans le sol de fondation est utilisé pour analyser les plaques circulaires sur les fondations élastiques. Les équations différentielles régissant le cas du chargement axisymétrique sont résolues pour deux variantes du modèle: les modèles de type Winkler et Pasternak. Des solutions sous forme fermée sont présentées pour différentes conditions de chargement sur de petites et grandes plaques circulaires. Le logiciel à éléments finis PLAXIS 2D est utilisé pour calibrer les variantes de modèle présentées en recherchant un accord en ce qui concerne la déflexion maximale. Des exemples numériques sont résolus pour démontrer les performances des modèles. Les résultats montrent que les modèles proposés conviennent pour effectuer efficacement des analyses de plaques et de poutres.

1 INTRODUCTION

Regular use of plates in civil and mechanical engineering practice initiated the development of analytical methods to analyze foundations interacting with the underlying soil. The earliest soil-structure-interaction (SSI) model devised for this purpose is Winkler's model (Winkler, 1867) that assumes a simple linear relationship between the contact pressure and the displacement at a point. This rudimentary model survived the test of time and technological advancement even though it is understood that the model is highly simplistic and far from representing the reality. One of its obvious shortcomings is the omission of the soil shear in the SSI. Despite this fact, structural analysis software still use Winkler's model to account for SSI effects (Worku, 2013).

Efforts to improve on the known drawbacks of Winkler's model have been underway following two approaches: elastic continuum and mechanical. The elastic continuum approach idealizes the subgrade as an elastic layer of thickness *H* overlying a rigid base characterized by the elastic parameters. However, almost all models are based on certain simplifying assumptions regarding the subgrade stresses, displacements or a combination of these to simplify the rather involved mathematical relationships Horvath (1983, 2002). For this reason, it makes sense to refer to these models as simplified continuum models.

The mechanical modelling approach, on the other hand, idealizes the subgrade as an assemblage of few mechanical elements such as springs, plates in pure shear or pure bending, stretched membranes and similar others. To this effect, improved models introduced additional mechanical elements to interconnect the springs and thereby overcome the inherent lack of shear interaction among the individual springs of Winkler's model (Filonenko-Borodich, 1950; Hetenyi, 1950; Pasternak, 1954; Kerr, 1964; Winkler, 1867).

Synthesis of the two approaches has the advantage to quantify the mechanical model parameters in terms of the known parameters of the continuum model while enabling the incorporation of the mechanical subgrade models in structural models.

Recently, the senior author proposed a generalized continuum subgrade model, unlike in previous continuum models, without neglecting any stress, strain and displacement components (Worku, 2010). This paper demonstrates the convenience and effectiveness of this model by analyzing circular plates of large and small radii under different loading conditions. It presents closed-form subgrade model parameters for circular plates adjusted by means of FEM models. Proposed values of the single adjustment factor in the model parameters is also provided for axisymmetric loading.

2 FORMULATION OF GOVERNING EQUATIONS

Equilibrium requirement for linear bending of isotropic, thin and solid circular plates with axisymmetric conditions on an elastic foundation results in the differential equation (DE) of

$$\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_o}{dr}\right)\right]\right\}+p(r)=q(r) \qquad [1]$$

Where $w_o(r)$ = vertical plate deflection; $q(r)$ = distributed vertical load; $p(r)$ = reaction of the elastic foundation and, $D = E_p h_p^3 / 12(1-v_p^2)$, flexural rigidity of the plate in which E_p = modulus of elasticity of the circular plate and h_p = thickness of the plate (Reddy, 2007; Szilard, 2004).

The final form of this DE is obtained after introducing the respective mathematical models as provided below.

2.1 Single-Parameter (Winkler's) Subgrade Model

In this model, the plate-soil interaction is represented by the highly simplified linear algebraic relationship between the contact pressure and the vertical deflection as

$$p(r) = k_w w_o(r) \qquad [2]$$

Where k_w = Winkler's coefficient of subgrade reaction.

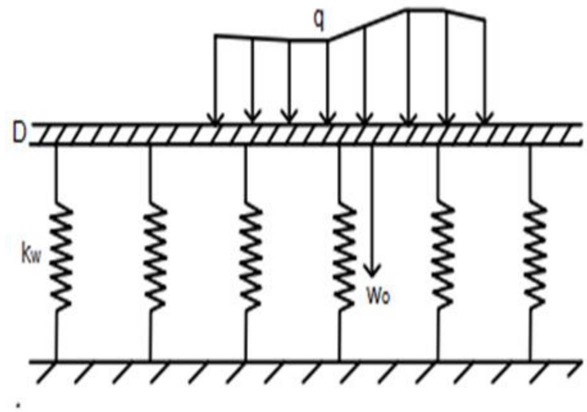

Figure 1. A Circular plate on Winkler's mechanical mode

Substituting Eq. 2 into equation 1, the governing DE takes the form

$$\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_o}{dr}\right)\right]\right\}+k_w w_o(r)=q(r) \qquad [3]$$

2.2 Two-Parameter (Pasternak's) Subgrade Model

A pure shear layer on top of Winkler's model is introduced to interconnect the springs at their heads thereby improving the shear interaction.

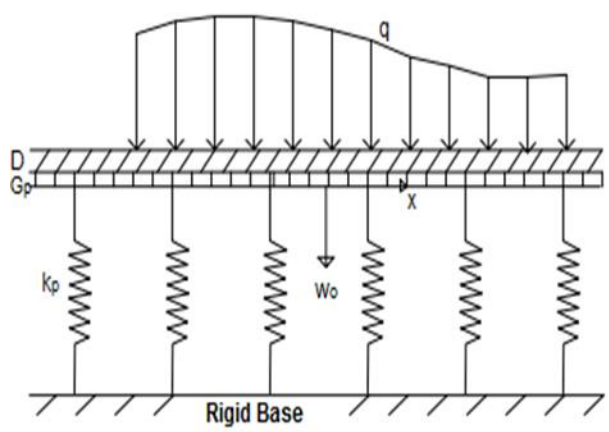

Figure 2. A Circular plate on Pasternak's mechanical model

The reaction of the elastic foundation is given by

$$p(r) = k_p w_o(r) - G_p \nabla^2 w_o(r) \qquad [4]$$

Where ∇^2 = laplacian operator; k_p = spring coefficient per unit area; G_p = coefficient of the shear element in Pasternak's model with the dimension of force per unit length (Pasternak, 1954).

Substituting Eq. 4 into equation 1, the governing DE becomes

$$\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw_o}{dr}\right)\right]\right\}+k_p w_o\left(r\right) \\ -\frac{G_p}{r}\frac{d}{dr}\left(r\frac{dw_o}{dr}\right)=q\left(r\right) \quad [5]$$

3 ANALYTICAL SOLUTION

3.1 Plates on a Winkler Subgrade Model

The circular plate is assumed to be subjected to a symmetrically distributed loading with respect to the center. The homogeneous form of Eq. 3 may be rewritten as

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w_o}{dr^2}+\frac{1}{r}\frac{dw_o}{dr}\right)+\frac{k_w w_o}{D}=0 \quad [6]$$

Eq. 6 reduces to

$$\Delta_r{}^2 w_o+\lambda^4 w_o=0 \quad [7]$$

Where $\lambda=\sqrt[4]{k_w/D}$ is the characteristic size of the circular plate-soil system (Hetenyi, 1979) that relates the subgrade and plate rigidities; $\Delta_r=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$ is the differential operator in polar coordinate for axisymmetric conditions.

Rigorous solutions of a DE of this type can be derived in form of power series, which may be expressed in terms of Bessel functions (Boas, 1966; Hildebrand, 1962).

The general solution of Eq. 7 is given by

$$w_o=A_1 z_1\left(\lambda r\right)+A_2 z_2\left(\lambda r\right)+A_3 z_3\left(\lambda r\right)+A_4 z_4\left(\lambda r\right) \quad [8]$$

Where A_1 to A_4 are open constants and, $z_1(\lambda r)$, $z_2(\lambda r)$, $z_3(\lambda r)$ and $z_4(\lambda r)$ are four independent solutions of the governing equation with the argument λr. These functions are infinite series solutions of modified Bessel equations.

Eq. 8 can be used to solve the problem of different loading types by introducing the pertinent boundary conditions to solve the constants.

3.2 Plates on a Pasternak Subgrade Model

Following similar steps as above, the homogeneous form of Eq. 5 can be expressed as

$$\Delta_r{}^4 w_o\left(r\right)-\frac{G_p}{D}\Delta_r{}^2 w_o\left(r\right)+\frac{k_p}{D}w_o\left(r\right)=0 \quad [9]$$

Where, the following substitutions are made:

$$1/l=\lambda; G_p/D=2T/l^2; k_p/D=1/l^4 \quad [10a]$$

In which

$$T=\sqrt{G_p{}^2/4k_p D} \quad [10b]$$

Equations of the form of Eq. 9 have a characteristic polynomial equation of the same order, the roots of which give the argument of the modified Bessel functions (Spiegel, 1971). This polynomial has the form:

$$d^4-2\frac{T}{l^2}d^2+\frac{1}{l^4}=0 \quad [11a]$$

The roots are given by

$$d_{1,2,3,4}=\pm\frac{1}{l}\sqrt{T\pm\sqrt{T^2-1}} \quad [11b]$$

There are three possible solutions depending on whether $T>1$, $T=1$ or $T<1$. Of practical significance is only the case of $T>1$. The solution for this case becomes

$$w_o\left(r\right)=A_1 J_o\left(\sqrt{d}r/l\right)+A_2 H_o^{(1)}\left(\sqrt{d}r/l\right) \\ +A_3 J_o\left(\sqrt{\bar{d}}r/l\right)+A_4 H_o^{(2)}\left(\sqrt{\bar{d}}r/l\right) \quad [12]$$

Where A_1 to A_4 are open constants, $J_o\left(\sqrt{d}r/l\right)$ & $J_o\left(\sqrt{\bar{d}}r/l\right)$ and $H_o^{(1)}$ & $H_o^{(2)}$ are Bessel functions of first and third kind resp. and, d and $\bar{d}$ are introduced constants for the root expressed as

$$d = \frac{l^2}{2}\left[-\frac{G_p}{D} + \sqrt{\left(\frac{G_p}{D}\right)^2 - \frac{4k_p}{D}}\right]$$
$$\bar{d} = \frac{l^2}{2}\left[-\frac{G_p}{D} - \sqrt{\left(\frac{G_p}{D}\right)^2 - \frac{4k_p}{D}}\right] \quad [13]$$

4 NUMERICAL ANALYSIS AND CALIBRATION

As stated earlier, the synthesis of continuum and mechanical subgrade models of the same order enables one to express the unknown mechanical model parameters in terms of the elastic constants of the soil layer and its thickness. Accordingly, the relations are obtained from the two models proposed by Worku (2013, 2014) as given below.

Winkler's model parameter:

$$k_w = \frac{E_s}{\left(1 - 0.4v_s\right)H} \quad [14]$$

Pasternak's model parameters:

$$k_p = \frac{\left(0.4v_s + 0.67\right)E_s}{H}, G_p = \left(1.36v_s + 2.28\right)GH \quad [15]$$

Where E_s = modulus of elasticity of the soil; v_s = Poisson's ratio of the soil; H = thickness of the soil layer; G = shear modulus of the elastic medium.

4.1 Identification of the important solution case for Pasternak's subgrade model

As pointed above, for Pasternak's subgrade model the case of $T > 1$ is the only case that has practical significance. This observation can be made by plotting T in Eq. 10 against a dimensionless stiffness factor or relative rigidity of the soil-plate system, k_r, as suggested by Rajapakse and Selvadurai (1991).

$$K_r = \frac{E_p\left(\frac{h_p}{a}\right)^3}{G} \quad [16]$$

Where a = radius of the plate; E_p = modulus of elasticity of the circular plate and h_p = thickness of the plate.

A range of soil properties that is wide enough to capture all possible cases is considered to establish the trend of T with k_r. These are given in Table 1 along with the plate properties.

Table 1. Soil and Plate Properties (Bowles, 1997; Das, 2007)

Soil parameters			Plate Property	
	E_s (KN/m^2)	μ_s	Plate type	RC
Soft Clay	15,000	0.4	E_p (Gpa)	25
Medium Stiff Clay	30,000	0.3	v_p	0.2
Stiff Clay	80,000	0.25	h_p (m)	0.15
Loose sand	20,000	0.3	ϕ (m)	20
Medium Dense Sand	40,000	0.25		
Dense Sand	81,000	0.2		

The plot of T against k_r for various values of stratum thickness normalized with respect to the radius of the plate is shown in Figure 3. It shows that T is almost always larger than one, exceptions being very small rigid plates on thin very soft soils. The odds for such a case to occur in practice are rare. Thus, the case $T > 1$ alone is pursued further.

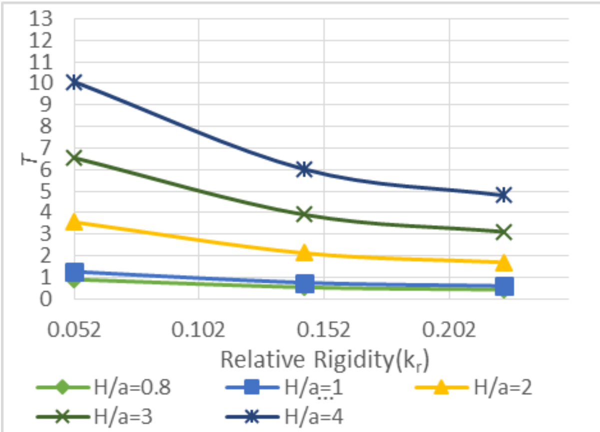

Figure 3. Effects of k_r and H on T

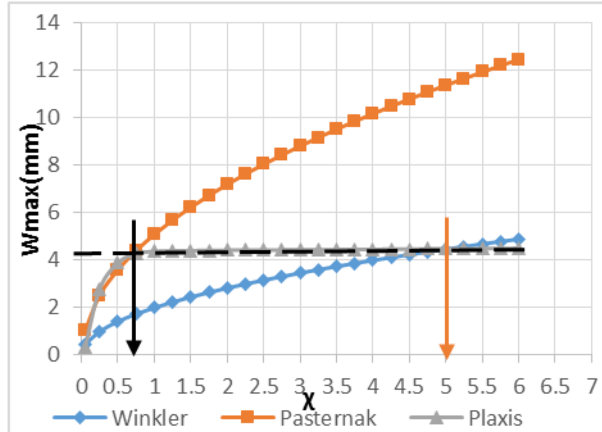

Figure 4. Determination of χ using central concentrated load and uniformly distributed load for large circular plates.

4.2 Determination of the Calibration Factor

All continuum models based on an elastic stratum are sensitive to the stratum thickness, H. This happens regardless of the degree of rigor of the models including the proposed Winkler-type and Pasternak-type model variants of Worku (2013, 2014). In order to overcome this problem, H is replaced by χB, where B is the foundation characteristic width and χ is used as a calibration factor to adjust selected responses against measured data, if available, or against rigorous numerical models. Plaxis 2D is used in this work for this purpose.

To undertake the calibration, an optimum mesh size is obtained first. After fixing the mesh size follows calibration of both models and a selected soil type is taken for various relative thickness of stratum, $\chi = H/\phi$, where ϕ is the diameter of the circular plate. The maximum deflection in mm is plotted against the dimensionless calibration factor, χ as shown in Figures 4 and 5, where only typical graphs are presented. The value of χ that gives the same value as the asymptotic value of the PLAXIS plot is identified. This is repeated for a range of cases with corresponding values of λ. The plot of χ obtained in this manner against a sufficient range of λ is shown in Figures 6 and 7 for the two proposed models.

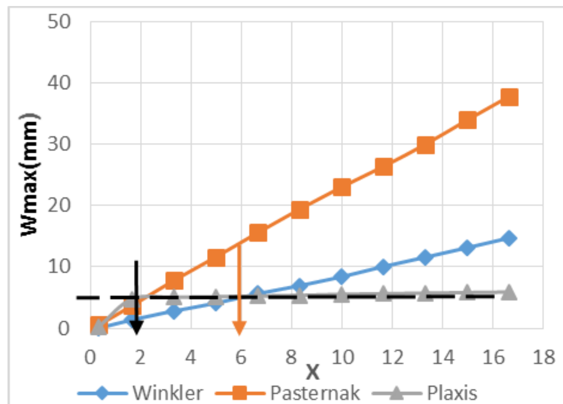

Figure 5. Determination of χ using central concentrated load and circumferential edge load for small circular plates.

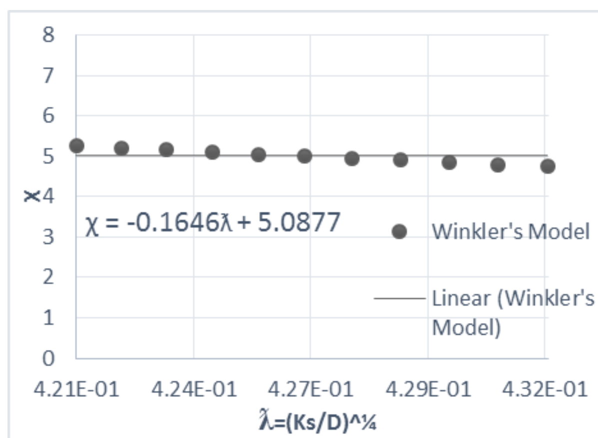

Figure 6. Calibration of Winkler's model for a central concentrated loading on large circular plates on soft clay.

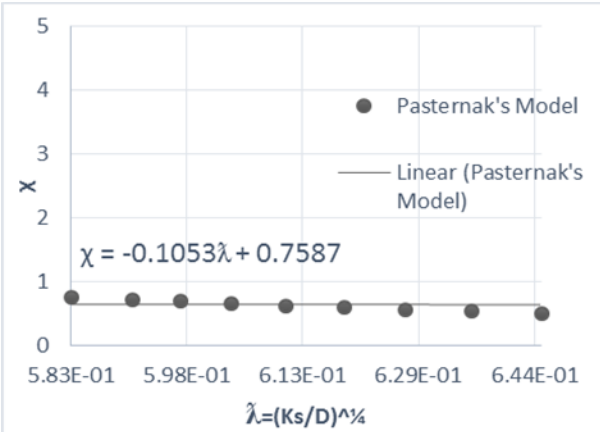

Figure 7. Calibration of Pasternak's model for a central concentrated loading on large circular plates on soft clay.

Though not presented here, similar plots for a distributed and circumferential edge load on diverse soil types were obtained. After observing all results, the recommended χ values for large and small circular plates on weak to strong soils are summarized in Tables 2 and 3.

Table 1. Recommended values of χ for large circular plates

Soil Type	Central Concentrated Load		Uniformly Distributed Load	
	Winkler's Model (χ_w)	Pasternak's Model (χ_p)	Winkler's Model (χ_w)	Pasternak's Model (χ_p)
Weak soils	5.03	0.68	5.76	1.21
All other soils	5.0	0.63	5.71	1.15

Table 3. Recommended values of χ for small circular plates

Soil Type	Central Concentrated Load		Circumferential Edge Load	
	Winkler's Model (χ_w)	Pasternak's Model (χ_p)	Winkler's Model (χ_w)	Pasternak's Model (χ_p)
Weak soils	6.01	2.37	6.47	2.36
All other soils	5.98	2.35	6.40	2.24

Hence, for an actual stratum with $H > \chi\phi$, it is recommended to take the calibration factors given in Tables 2 and 3. If $H \leq \chi\phi$, the actual depth may be taken without modification.

4.3 Calibration of Model Parameters

The sensitivity of the deflection to the layer thickness can be avoided by introducing the following relation

$$H = \chi\phi \qquad [17]$$

Where χ = calibration factor established above for both models. Inserting Eq. 17 into equation 14 and 15, the calibrated model parameter for Winkler's subgrade model becomes

$$k_w = \frac{E_s}{(1 - 0.4v)\chi_w\phi} \qquad [18]$$

Similarly, for Pasternak's subgrade model, the parameters take the form

$$k_p = \frac{(0.4v + 0.67)E_s}{\chi_p\phi},\ G_p = (1.36v + 2.28)G\chi_p\phi \qquad [19]$$

4.4 Performance of the Calibrated Models

The calibrated Winkler-type and Pasternak-type models are compared with PLAXIS 2D results for selected loading conditions and for circular plates of large and small radii. The plate is classified as large, intermediate and small adopting the classification of beams proposed by Hetenyi (1979) with some adjustments.

i. Small circular plates: $\lambda\phi < \pi / 4$
ii. Intermediate circular plates: $\pi / 4 < \lambda\phi < \pi$
iii. Large circular plates: $\lambda\phi > \pi$

Circular plates which satisfy the requirements of intermediate plates can be classified and treated as small plates according to Selvadurai (1979).

4.4.1 Large Plates

Numerical examples are solved for large plates subjected to loads concentrated at the center and to uniformly distributed loading. The deflection curves are presented in Figures 11 to 14 for the soil parameters presented in Table 1, plate properties in Table 4 and recommended χ values in Table 2. Similarly, deflection plots for both loading conditions on diverse soils types were obtained though not presented here. In advance to these plots, for the same soil and plate data sets, typical Plaxis outputs of a circular plate subjected to a central concentrated load are presented in Figures 8 to 10. While modelling of the circular plate in Plaxis 2D, linear elastic and axisymmetric material models are assumed for the soil and plate respectively.

Table 4. Calculation cases for large plates

Plate Dimension and Properties		Soil Thickness (H)	Loading	
Modulus of elasticity, E_p	25 (GPa)	120(m)	Vertical concentrated, p	100 (KN)
Poisson's ratio, v_p	0.2		Uniformly distributed, q	40 (KN/m^2)
Thickness, h_p	0.15		Radius of loaded region, a	5(m)
Plate diameter, ϕ	20(m)			

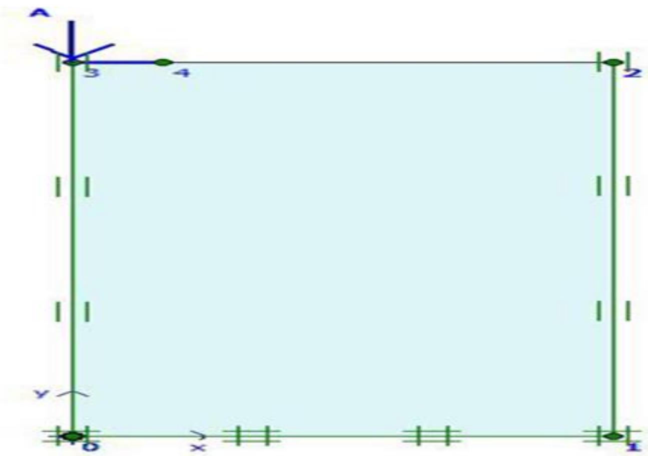

Figure 8. Geometry Model

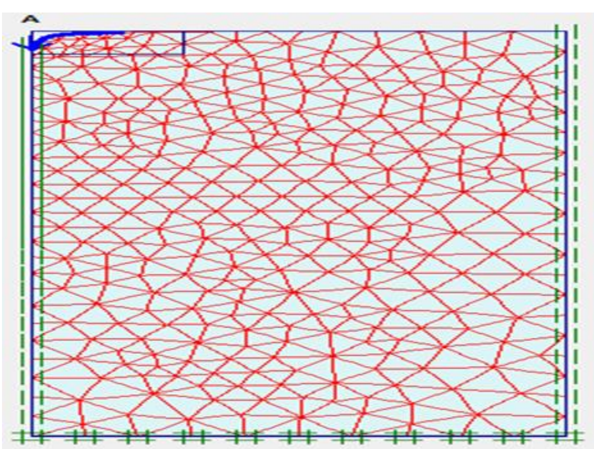

Figure 9. Deformed mesh of a circular plate on stiff clay subjected to central concentrated load

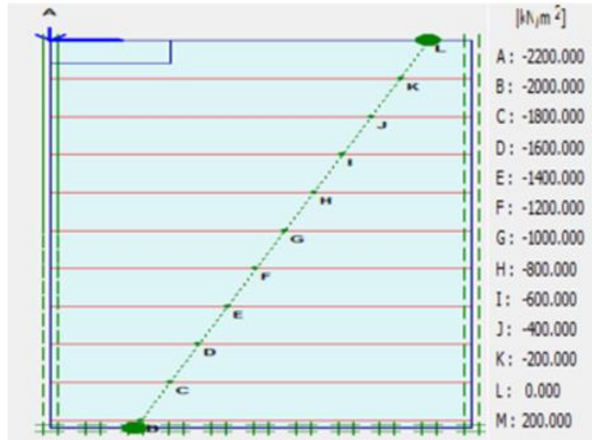

Figure 10. Contour line representation of the effective mean stress for a stiff clay soil

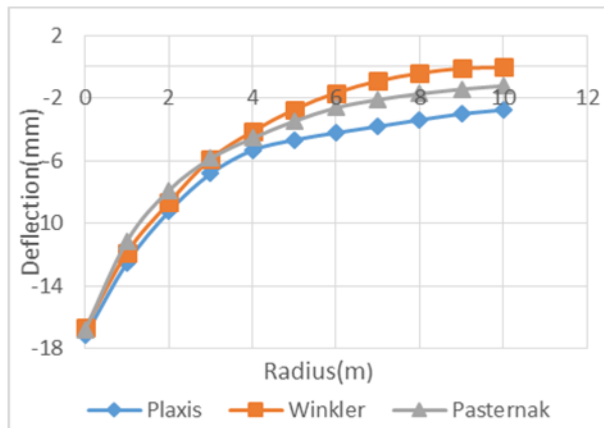

Figure 11. Deflection of a large circular plate on soft clay subjected to central concentrated load

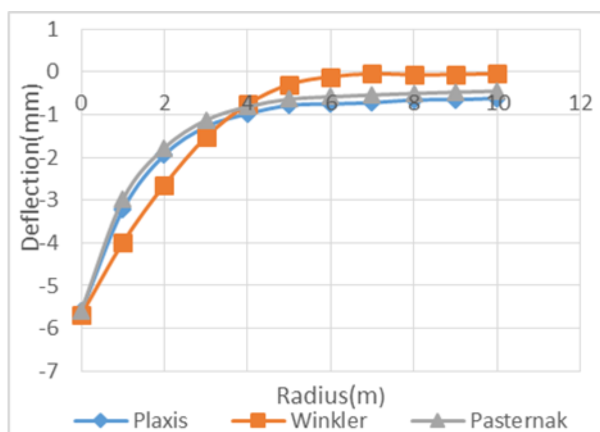

Figure 12. Deflection of a large circular plate on stiff clay/dense sand subjected to central concentrated load

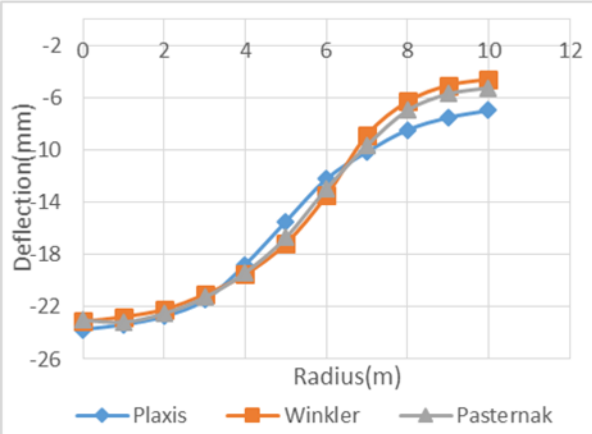

Figure 13. Deflection of a large circular plate on soft clay subjected to uniformly distributed load

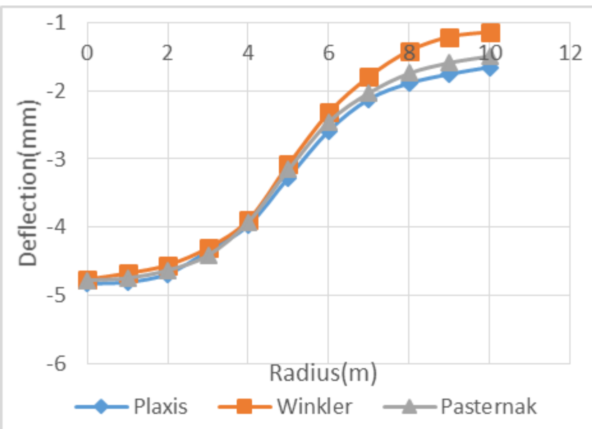

Figure 14. Deflection of a large circular plate on stiff clay/dense sand subjected to uniformly distributed load

Figures 11 to 14 show that the use of the recommended calibration factors give maximum deflections in very good agreement with that of the FE result for both models. A relatively visible discrepancy is observed in weak soils when Winkler's model is used. However, this is of little practical significance indicating that even Winkler's model can yield satisfactory results if properly calibrated. Pasternak's model performs even much better over the whole area of the plates as evidenced by a wider range of analysis results not reported here.

4.4.2 Small Plates

Similarly, small plates subjected to a concentrated load at center and a circumferential load at the edge are analyzed with the soil parameters and the recommended χ values presented in Tables 1 and 3. The plate size and loadings are given in Table 5. Deflection curves on weak and strong soils are given in Figure 15 to 18.

Table 5. Calculation cases for small plates

Plate Dimension and Properties		Soil Thickness (H)	Loading	
Modulus of elasticity, E_p	25 (GPa)	21(m)	Vertical concentrated, p_o	100 (KN)
Poisson's ratio, v_p	0.2		Edge load, q_o	100 (KN/m)
Thickness, h_p	0.15			
Plate diameter, ϕ	3(m)			

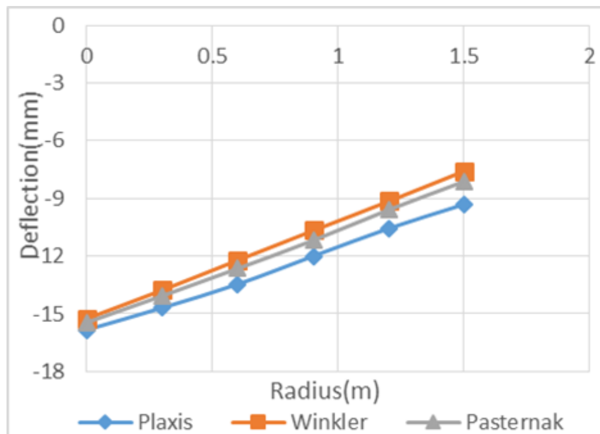

Figure 15. Deflection of a small circular plate on soft clay subjected to central concentrated load

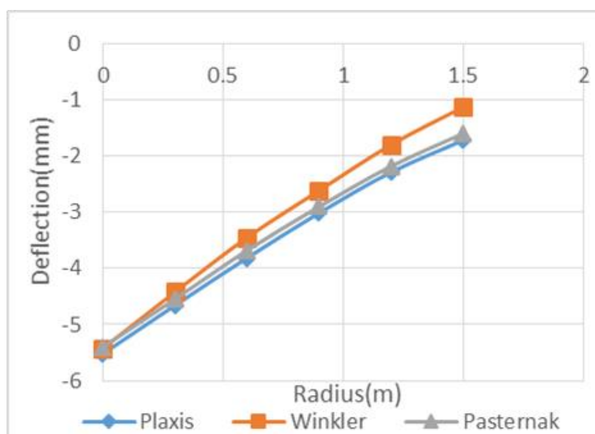

Figure 16. Deflection of a small circular plate on stiff clay/dense sand subjected to central concentrated load

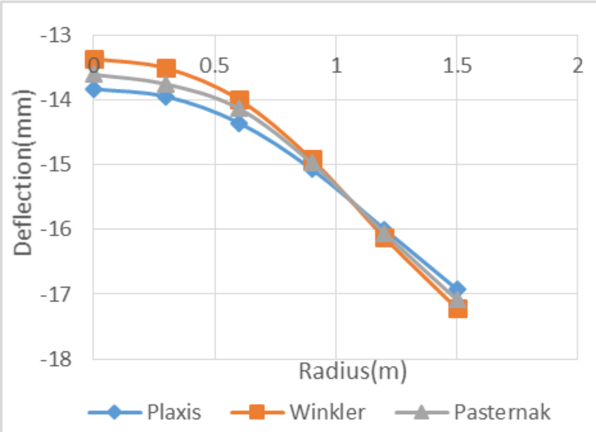

Figure 17. Deflection of a small circular plate on soft clay subjected to circumferential edge load

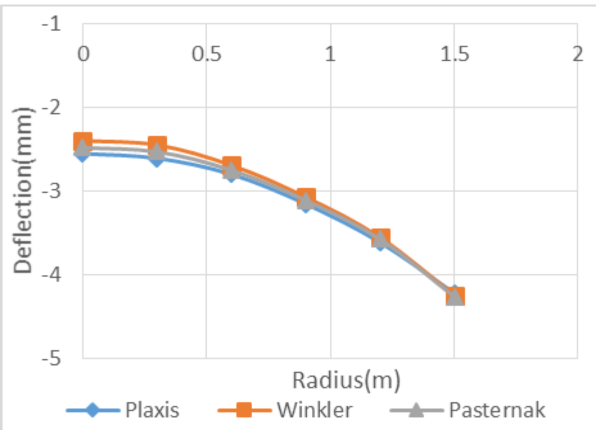

Figure 18. Deflection of a small circular plate on stiff clay/dense sand subjected to circumferential edge load

All plots show very good agreements of both models with the PLAXIS 2D, even though the maximum deflections are slightly underestimated by Winkler's model in the case of the centrally loaded plates.

The deviations in the internal moments and shear forces, though not presented here, are generally negligible.

5 CONCLUSION

The potential use of the generalized continuum-based subgrade model proposed by the senior author in the analysis of axisymmetric circular plates on an elastic foundation is demonstrated. The inherent sensitivity of the both model variants to the layer thickness has been avoided by calibrating them with respect to the thickness itself normalized with respect to the plate diameter. The small discrepancies observed with respect to the FE results are of little practical significance, especially if one notes that the calibration is based on matching only the maximum deflection. Treating a small plate with a circumferential edge loading using the respective calibration factors has shown similar trend of the results as in the other loading conditions.

Values of the calibration factor obtained for Winkler's model is much larger than that for Pasternak's model. This is indicative of the significant contribution of the soil shear in the plate response. It also shows that the use of unadjusted model parameters can lead to flawed results. This is particularly important when one recalls that there is a plethora of relationships for Winkler's subgrade modulus in the literature.

The study shows that the calibrated Pasternak model represents the real physical problem much better than the calibrated Winkler model for it directly accounts for the shear interaction missing in Winkler's model. Hence, the study shows that the proposed rigorous Kerr-Equivalent Pasternak model is attractive and more appropriate for the analysis of plates and beams on elastic foundations.

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