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Differential Settlement of Foundations

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ABSTRACT
The allowable settlement of a foundation must account for maximum deformations at both serviceability limit states (SLS) and ultimate limit states (ULS) of the supported structure. The resulting design of foundations typically proceeds by limiting the total settlement of individual foundations and by doing so, hopefully restricting the differential settlement between pairs of foundations. In fact, it is often the differential settlement which is of more importance to the performance of the supported structure. Unfortunately, due to the random nature of the supporting ground, the magnitude of differential settlement is also random, and is typically much more difficult to characterize than is the total settlement of individual foundations. For a group of four foundations arranged in a square pattern, this paper investigates the distribution of the pair-wise maximum differential settlement as a function of the degree of spatial variability and correlation. In particular, the effect of increasing correlation length between elasticity of the ground, and loads applied to the foundations, on the maximum differential settlement distribution is investigated. The end goal is to develop probabilistic design requirements which allow for acceptable target reliabilities against excessive differential settlement.

RÉSUMÉ
Le tassement admissible d'une fondation doit tenir compte des déformations maximales aux états limites de service (SLS) et ultimes (ULS) de la structure supportée. La conception des fondations qui en résulte procède généralement en limitant le tassement total des fondations individuelles et, ce faisant, en limitant, espérons-le, le tassement différentiel entre les paires de fondations. En fait, c'est souvent le règlement différentiel qui est plus important pour la performance de la structure supportée. Malheureusement, en raison de la nature aléatoire du sol de support, l'amplitude du tassement différentiel est également aléatoire et est généralement beaucoup plus difficile à caractériser que le tassement total des fondations individuelles. Pour un groupe de quatre fondations disposées en carré, cet article étudie la distribution du tassement différentiel maximum par paire en fonction du degré de variabilité spatiale et de corrélation. En particulier, l'effet de l'augmentation de la longueur de corrélation entre les élasticité du sol et les charges appliquées aux fondations sur la distribution de tassement différentielle maximale est étudié. L'objectif final est d'élaborer des exigences de conception probabilistes qui permettent des fiabilités cibles acceptables contre un tassement différentiel excessif.

1 INTRODUCTION

Whenever multiple foundations are used to support a structure, the potential for differential settlement between the foundations often governs the foundation design. In practice, the maximum differential settlement between pairs of foundations is usually unknown, and it is the total settlement of individual foundations that is used in the design process. Generally speaking, the maximum differential settlement is usually assumed to be some fraction, e.g. one half, of the total settlement of individual foundations. However, it would be beneficial to characterize the distribution of maximum differential settlement for use in design.

The probabilistic analysis of foundation settlements has received some attention over the years. Zeitoun and Baker (1992) examined differential settlements between pairs of foundations to the problem of two beams acting on three supports. Roberts and Misra (2009) developed a reliability-based design approach for deep foundations which considered differential settlement between a pair of deep foundations. Naghibi et. al (2016) also investigated the differential settlement between a pair of foundations and developed a distribution of the differential settlement which was validated by Monte-Carlo simulation of a pair of foundations in a spatially random soil. In the current paper, the approach is extended to a group of four foundations, as depicted in Figure 1, and the distribution of the maximum differential settlement between the foundations is studied. The foundation group is assumed to be placed on a spatially variable linearly elastic soil and the loads applied to each foundation are random and cross-correlated to some extent. Although soil deformations are well-known to be non-linear, the extension of this study to a non-linear elastic-plastic soil is a much more ambitious numerical problem, especially when random fields are included. Such an analysis is left for future study. Note that the effective properties of the elastic soil will be assumed here to be the linearized response of the soil in the region of the mean soil displacements.

Using Monte Carlo simulation, realizations of the effective elastic moduli (expressed as a spring constant) under each foundation along with the applied load are used to predict the settlement of each foundation, from which the maximum differential settlement is extracted. Each simulation involves the generation of spring constant and load values that are cross-correlated according to separation distance between foundations.

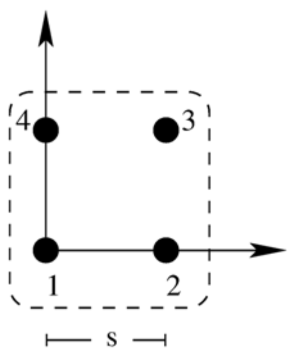

Figure 1: plan view of foundation locations

The paper is organized as follows: In Section 2, a random field model is presented for a system of n_p foundations. The corresponding simulation model is described in Section 3. The methodology is discussed in Section 4 and the results are presented in Section 5. An example of how the results of this paper can be used to estimate probabilities of excessive differential settlement is given in Section 6. Finally, conclusions and suggestions for future work are in Section 7.

2 RANDOM FIELD MODEL

A random field $X(\underset{\sim}{x})$ is a collection of random variables $X_1 = X(\underset{\sim}{x}_1), X_2 = X(\underset{\sim}{x}_2), \ldots$, whose values are associated with each spatial location $\underset{\sim}{x}$. Values in a random field are usually spatially correlated, and the spatial dependence in a field is characterized by the field correlation structure, which is commonly specified through a correlation function parameterized by correlation length, θ. In this paper, an isotropic exponentially decaying Markov correlation function is used, defined by

$$\rho(\tau_{ij}) = \exp\left\{\frac{-2|\tau_{ij}|}{\theta}\right\} \qquad [1]$$

where τ_{ij} is the distance between any two points, X_i and X_j, in the field, and θ is the correlation length (Fenton and Griffiths, 2008).

A lognormal distribution is commonly used for modeling engineering properties due to its non-negative nature and its simple relationship with the normal distribution. In particular, a lognormal random field can be easily produced through a simple transformation of a Gaussian random field. In general, if X is lognormal with mean and standard deviation μ_X and σ_X, then $\ln X$ is normal with parameters

$$\begin{aligned} \sigma^2_{\ln X} &= \ln(1+v_X^2) \\ \mu_{\ln X} &= \ln(\mu_X) - \frac{1}{2}\sigma^2_{\ln X} \end{aligned} \qquad [2]$$

where $v_X = \sigma_X / \mu_X$ is the coefficient of variation of X. In this research, both load, F, and spring constant, K, are assumed to be lognormally distributed random variables. This implies that $\ln F$ and $\ln K$ are both normally distributed with parameters given by Eq. 2 (where the subscript X is suitably replaced by either K or F). Furthermore, both load and spring constant are spatially varying random variables with an additional parameter being the correlation length, $\theta_{\ln F}$ or $\theta_{\ln K}$ respectively, replacing θ in Eq. 1. The spring constant K would normally be derived from the effective elastic modulus of the soil taking into account the geometry of the foundation and the thickness of the supporting foundation layer (see, e.g., Poulos and Davis, 1974, and Fenton and Griffiths, 2005).

3 SIMULATION MODEL

Various random field generation algorithms exist of which the Covariance Matrix Decomposition (CMD, see e.g., Fenton and Griffiths, 2008) method is employed in this research to provide realizations of the random load and spring constant fields. CMD is an exact method of producing realizations of a discrete random field (i.e., at the foundation locations) which requires the mean, $\mu_{\ln X}$, and covariance matrix, $\underset{\approx}{C}$, having elements $C_{ij} = \rho_{ij}\sigma_{\ln X_i}\sigma_{\ln X_j}$, $i, j = 1, 2, \ldots, n_p$. The elements of $\underset{\approx}{C}$ are the covariances between any pair of points in the field separated by lag distance τ_{ij}, where $\rho_{ij} = \rho(\tau_{ij})$ (see Eq. 1). For a stationary random field, having spatially constant variance, the covariance matrix $\underset{\approx}{C}$ is composed of the elements

$$C_{ij} = \begin{cases} \sigma^2_{\ln X} & i = j \\ \sigma^2_{\ln X}\rho_{ij} & i \neq j \end{cases} \quad [3]$$

Since $\underset{\approx}{C}$ is a positive definite matrix, then a properly correlated normally distributed (Gaussian) random field $G_i = G(\underset{\sim}{x}_i)$, where $\underset{\sim}{x}_i$ is a point in the field, can be produced according to

$$\underset{\sim}{G} = \underset{\sim}{\mu}_{\ln X} + \underset{\approx}{L}\underset{\sim}{Z} \quad [4]$$

where $\underset{\approx}{L}$ is a lower triangular matrix satisfying $\underset{\approx}{L}\underset{\approx}{L}^T = \underset{\approx}{C}$ (obtained using a Cholesky Decomposition), and $\underset{\sim}{Z}$ is a vector of n_p independent standard normal random variables (mean zero, unit variance).

The lognormal random field, $\underset{\sim}{X}$, is obtained from the normal field, $\underset{\sim}{G}$, using the following transformation:

$$\underset{\sim}{X} = \exp\{\underset{\sim}{G}\} \quad [5]$$

Although CMD is simple, it can be inefficient and inaccurate for large fields. For example, a field of size $n \times n$ requires a covariance matrix of size $n^2 \times n^2$ so that if n is large, the CMD becomes numerically intensive and unstable. However, CMD is perfectly adequate for small random fields, such as the one used in this research having a field size of 2×2, depicted in Figure 1.

4 METHODOLOGY

As mentioned previously, both the load and the soil's elastic spring constant are assumed to be lognormally distributed. An individual foundation is subjected to random load F_i having mean μ_F and standard deviation σ_F. The foundation is also supported by an elastic soil with random effective spring constant K_i having mean μ_K and standard deviation σ_K. For each realization of K_i and F_i, $i = 1, 2, ..., n_p$, where n_p is the number of foundations in a foundation group, the maximum differential settlement is defined as

$$\Delta_{\max} = \max_{i \neq j}\left|\delta_i - \delta_j\right| \quad [6]$$

where

$$\delta_i = \frac{F_i}{K_i} \quad [7]$$

is the settlement of foundation i. The settlement, which is the ratio of load over spring constant, is also random and lognormally distributed. Thus,

$$\ln\delta_i = \ln\left(\frac{F_i}{K_i}\right) = \ln F_i - \ln K_i \quad [8]$$

is normal with parameters:

$$\begin{aligned} \mu_{\ln\delta} &= \mu_{\ln F} - \mu_{\ln K} \\ \sigma^2_{\ln\delta} &= \sigma^2_{\ln F} + \sigma^2_{\ln K} \end{aligned} \quad [9]$$

where independence between the random variables F_i and K_i (or $\ln F_i$ and $\ln K_i$) was assumed in order to compute $\sigma^2_{\ln\delta}$. With reference to Eq. 2, the mean and variance of load, F_i, are

$$\begin{aligned} \sigma^2_{\ln F} &= \ln(1 + v_F^2) \\ \mu_{\ln_F} &= \ln(\mu_F) - \frac{1}{2}\sigma^2_{\ln F} \end{aligned} \quad [10]$$

Similarly,

$$\begin{aligned} \sigma^2_{\ln K} &= \ln(1 + v_K^2) \\ \mu_{\ln_K} &= \ln(\mu_K) - \frac{1}{2}\sigma^2_{\ln K} \end{aligned} \quad [11]$$

The simulation process is carried out as follows:

1. Two lognormal random fields, each of size n_p, are generated representing loads, F_i, and spring constants, K_i, associated with individual foundations in a foundation system arranged as depicted in Figure 1.

2. For each foundation, settlement is calculated using Eq. 7 and the maximum differential settlement is determined for the foundation group using Eq. 6.

3. Steps 1 and 2 are repeated n_{sim} times and the mean, $\mu_{\Delta_{\max}}$, and the standard deviation, $\sigma_{\Delta_{\max}}$, of the maximum differential settlement is calculated, for all cases listed in Table 1.

An exponential function of the form

$$\mu_{\Delta_{\max}} = a_1 + a_2 e^{a_3\theta} \quad [12]$$

was fit to $\mu_{\Delta_{\max}}$ by least squares regression, where

$$\begin{aligned} a_1 &= 0.3621 + 0.4492\,v \\ a_2 &= -0.1570 + 1.714\,v \\ a_3 &= -0.2440 + 0.1378\,v \end{aligned} \quad [13]$$

Similarly, $\sigma_{\Delta_{\max}}$ is predicted by

$$\sigma_{\Delta_{\max}} = b_1 + b_2 e^{b_3\theta} \quad [14]$$

where

$$b_1 = 0.1333 + 0.5595\,v$$
$$b_2 = -0.09564 + 0.9009\,v \qquad [15]$$
$$b_3 = -0.2406 + 0.2160\,v$$

For simplicity, the coefficients of variation and the correlation lengths of the random load and spring constant fields are assumed to be the same. In other words, it is assumed that $v_K = v$ and $\theta_{\ln K} = \theta$.

The probability that the maximum differential settlement is lower than some magnitude a can now be estimated using the lognormal distribution as follows:

$$\mathrm{P}[\Delta_{\max} < a] = \mathrm{P}[\ln \Delta_{\max} < \ln a] = \Phi\left(\frac{\ln a - \mu_{\ln \Delta_{\max}}}{\sigma_{\ln \Delta_{\max}}}\right) \qquad [16]$$

where $\mu_{\ln \Delta_{\max}}$ and $\sigma_{\ln \Delta_{\max}}$ are obtained through the transformations of Eq. 2 using the regressions given by Eq's. 12 and 14.

5 RESULTS AND DISCUSSION

The objective of this section is to estimate the distribution of the maximum differential settlement of a foundation group. The parameters used in this case study are listed in Table 1

Table 1. Input parameters used in simulation

Parameters	Values Considered
n_p	4
μ_F	1 N
μ_K	1 N/m
v_F	0.25
$v_K = v$	0.1 to 0.5
s	1 m
$\theta_{\ln F} / s$	2.0
$\theta_{\ln K} / s$	0.1 to 50.0
n_{sim}	1,000,000

The simulation involves $n_{sim} = 1{,}000{,}000$ realizations. The standard deviation of a probability estimate is thus $\sqrt{p(1-p)/n_{sim}} \simeq 0.001\sqrt{p}$ for small probability p. This means that if $p = 1\times 10^{-4}$, then the standard deviation of its estimate is about 10^{-5} and therefore, $n_{sim} = 1{,}000{,}000$ can accurately resolve probabilities down to about 10^{-4} and reasonably accurately down to about 10^{-5}.

To demonstrate the agreement between the regression prediction given by Eq's 12 and 14, the probability $\mathrm{P}[\Delta_{\max} < a]$ can be estimated both by the regression and by the simulation and the pair of probability estimates can be plotted against each other. In Figure 2, these pairs of probabilities are plotted for $a = 0.05, 0.10, 0.15, \ldots, 5.00$. The upper bound on a was selected to be approximately equal to the mean maximum differential settlement plus four standard deviations. For the worst case, when $v = 0.5$, $\theta = 0.1$, the mean maximum differential settlement is $\mu_{\Delta_{\max}} = 1.3$ with standard deviation $\sigma_{\Delta_{\max}} = 0.8$, resulting in a max value of $1.3 + 4(0.8) = 4.5$, which was rounded up to 5.00.

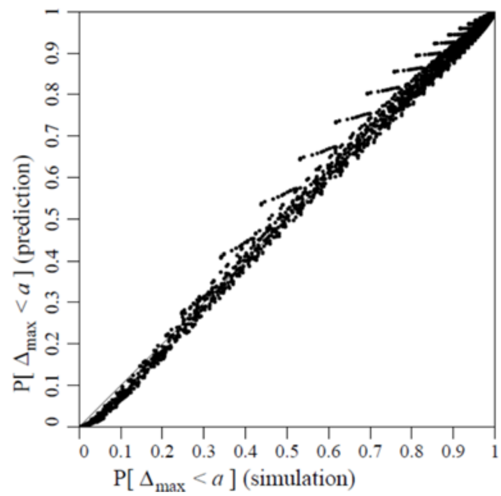

Figure 2. Predicted, obtained via Eq. [16], versus simulated probability $\mathrm{P}\left[\Delta_{\max} < a\right]$, for all cases listed in Table 1 and for $a = 0.05, 0.10, 0.15, \ldots, 5.00$.

Good agreement between the regression and simulation results would lead to all points lying along a line having slope 1. As can be seen from Figure 2, some of the regression results show trends that are slightly different from the simulation results. As it turns out, the poorer agreements occur for the lowest coefficient of variation considered, $v = 0.1$.

The reason for the poorer performance of the $v = 0.1$ regression lies in its use of the lognormal distribution. The lognormal distribution does not closely describe the actual distribution at this small coefficient of variation. Figure 3 compares a lognormal fit and a Gamma distribution fit to the actual frequency density. As can be seen, the Gamma distribution shows a much better agreement with the simulation results. The consideration of the Gamma distribution fit is left for future research and not pursued further in this paper since the lognormal does very well at the higher coefficients of variations and reasonably well even at $v = 0.1$.

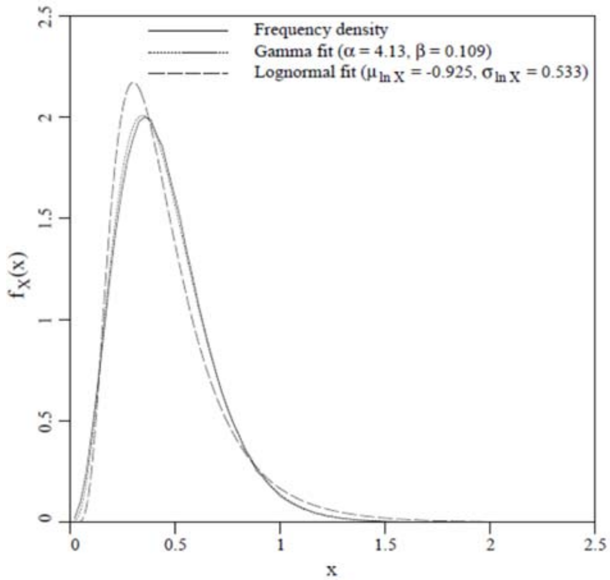

Figure 3: Frequency density of $X = \Delta_{\max}$ for $v = 0.1$ and $\theta_{\ln K} / s = 2.0$ with fitted Gamma and lognormal distributions.

To investigate how the regression derived probabilities perform at higher maximum differential settlements, Figure 4 has been produced to compare $\mathrm{P}[\Delta_{\max} > a]$ for higher values of $a = 3.00, 3.05, 3.10, \ldots, 5.00$. It can be seen that even at these relatively small probabilities, the agreement between predicted and simulated is reasonable.

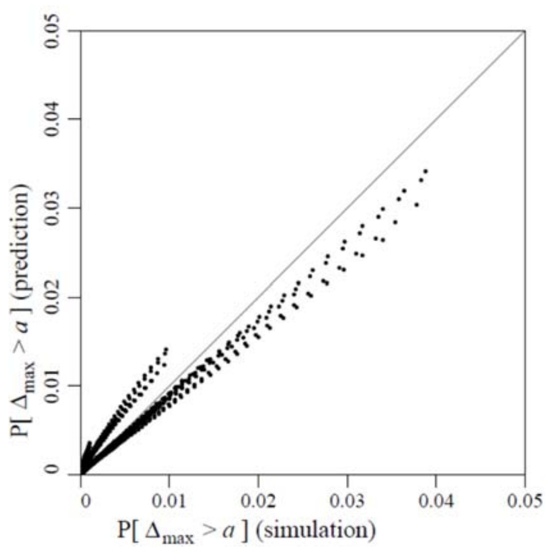

Figure 4. Predicted, obtained via Eq. [16], versus simulated probability $\mathrm{P}[\Delta_{\max} > a]$, for all cases listed in Table 1, and for $a = 3.00, 3.05, \ldots, 5.00$.

6 EXAMPLE

To illustrate how the results presented in this paper can be used in a design context, consider the following example. Suppose we have a 4-foundation system arranged on a square grid with center-to-center separation distance equal to $s = 5$ m, each subjected to a mean load $\mu_F = 200$ kN with coefficient of variation of $v_F = 0.25$. As in the above theory, it is assumed that the load correlation length is equal to twice the center-to-center foundation spacing ($\theta_{\ln F} / s = 2$). Suppose further that the supporting soil has mean effective spring constant $\mu_K = 20000$ kN/m , taking into account the geometry of the foundation, also with coefficient of variation of $v_K = 0.3$. Finally, assume that the correlation length of the soil is $\theta_{\ln K} = 15$ m.

The Canadian Foundation Engineering Manual (CGS, 2006) specifies allowable angular distortion of about 1/200 for structural damage (ULS) and 1/500 for cracking of walls and partitions (SLS). For a foundation spacing of $s = 5$ m, these limits correspond to maximum differential settlements, at least between closer foundations, of 5/200=0.025 and 5/500=0.01 for ULS and SLS respectively. Note that the diagonal distance has been ignored for simplicity. To handle this case properly, it would be better to look at maximum differential slope between pairs of foundations, which is left for future work.

In order to use the above regression results to find the failure probability $\mathrm{P}[\Delta_{\max} > a]$, the example values must be scaled to match the results presented in this paper, where $\frac{\mu_F}{\mu_K} = 1$ and $s = 1$. In other words, the maximum acceptable differential settlements must be scaled up by a factor of $\frac{\mu_F}{\mu_K} / \frac{200}{20000} = 100$ and the correlation length scaled down by a factor of $\frac{1}{s} = \frac{1}{5}$. This means that the acceptable maximum differential settlements become 100(0.025) = 2.5 for ULS and 100(0.01) = 1 for SLS. In addition, a correlation length of $\theta' = \frac{\theta_{\ln k}}{s} = \frac{15}{5} = 3$ should be used in calculating the mean and standard deviation of $\Delta_{\max}$.

Next, the values of a_1, a_2, a_3 can be calculated for $v = 0.3$ using Eq. 13,

$$a_1 = 0.3621 + 0.4492 \times 0.3 = 0.49686$$
$$a_2 = -0.1570 + 1.714 \times 0.3 = 0.3572$$
$$a_3 = -0.2440 + 0.1378 \times 0.3 = -0.20266$$

from which

$$\begin{aligned}\mu_{\Delta_{\max}} &= a_1 + a_2 e^{a_3 \theta'} \\ &= 0.49686 + 0.3572 \times \exp(-0.20266 \times 3) = 0.691337\end{aligned}$$

Similarly, for $v = 0.3$, Eq. 15 gives,

$$b_1 = 0.1333 + 0.5595 \times 0.3 = 0.30115$$
$$b_2 = -0.09564 + 0.9009 \times 0.3 = 0.17463$$
$$b_3 = -0.2406 + 0.2160 \times 0.3 = -0.1758$$

so that

$$\sigma_{\Delta_{\max}} = b_1 + b_2 e^{b_3\theta'}$$
$$= 0.30115 + 0.17463 \times \exp(-0.1758 \times 3) = 0.404206$$

The mean and standard deviation of $\Delta_{\max}$ are converted to log-space, using Eq. 2,

$$\sigma^2_{\ln\Delta_{\max}} = \ln\left(1+\nu^2_{\Delta_{\max}}\right) = \ln\left(1+\frac{0.404206^2}{0.691337^2}\right) = 0.294043$$

$$\mu_{\ln\Delta_{\max}} = \ln\left(\mu_{\Delta_{\max}}\right) - \frac{1}{2}\sigma^2_{\ln\Delta_{\max}}$$
$$= \ln(0.691337) - \frac{1}{2}\times 0.294043 = -0.51615$$

Finally, at ULS, the failure probability is computed as,

$$p_f = \mathrm{P}\left[\Delta_{\max} > 2.5\right] = \Phi\left(-\frac{\ln(2.5) - \mu_{\ln\Delta_{\max}}}{\sigma_{\ln\Delta_{\max}}}\right)$$
$$= \Phi\left(-\frac{\ln(2.5) - (-0.51615)}{\sqrt{0.294043}}\right) = \Phi(-2.641624) = 0.004$$

while at SLS,

$$p_f = \mathrm{P}\left[\Delta_{\max} > 1.0\right] = \Phi\left(-\frac{\ln(1.0) - \mu_{\ln\Delta_{\max}}}{\sigma_{\ln\Delta_{\max}}}\right)$$
$$= \Phi\left(-\frac{\ln(1.0) - (-0.51615)}{\sqrt{0.294043}}\right) = \Phi(-0.951852) = 0.17$$

If the target reliability at ULS is approximate 1/1000 (corresponding to a reliability index of about 3.0), then the foundation in this example is not safe enough for design at ULS. Similarly, the almost 17% failure probability at SLS seems excessively high in practice.

Note that the simulation run of this example, performed above, gives a ULS failure probability of 0.0014 (compared to the estimate above of 0.004), and an SLS failure probability of 0.15 (compared to the estimate above of 0.17). The agreement is quite reasonable. A difference of a factor of about 3 in the small ULS failure probability estimates is not particularly surprising. The important thing is that the probabilities are of the same order of magnitude.

7 CONCLUSIONS

In this paper, the distribution of the pair-wise maximum differential settlement of a 4-foundation group is investigated and developed as a function of the degree of spatial variability and correlation. The results of this paper assist in developing probabilistic design requirements which allow for acceptable target reliabilities against excessive differential settlement of foundations. A regression is derived for $\frac{\mu_F}{\mu_K} = 1$ and $s = 1$, which can be scaled to other foundation conditions to estimate ULS and SLS design failure probabilities, as demonstrated in Section 6.

This preliminary study only considers a set of 4 foundations arranged on a square grid. The study will be extended to more than 4 foundations and to the direct consideration of maximum differential slopes between foundations. Another extension considered for the future is to properly model the spatially variable soil and its interaction with the foundations using the random finite-element method (RFEM).

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LIST OF ABBREVIATIONS AND SYMBOLS USED

ULS	Ultimate Limit State
SLS	Serviceability Limit State
a_i	regression coefficients
b_i	regression coefficients
K	soil's spring constant (random)
F	true load (random)
n_p	number of foundations in a foundation system
n_{sim}	number of simulations
p_f	probability of failure $\left(=\mathrm{P}\left[\Delta_{\max} > a\right]\right)$
$\mathrm{P}[\cdot]$	probability operator
s	center-to-center foundation spacing

v common coefficient of variation of the random spring constant and load fields

v_K spring constant coefficient of variation $(=\sigma_K / \mu_K)$

v_F load coefficient of variation $(=\sigma_F / \mu_F)$

v_X coefficient of variation of the random field X $(=\sigma_X / \mu_X)$

$v_{\Delta_{\max}}$ coefficient of variation of the random field X $\left(=\sigma_{\Delta_{\max}} / \mu_{\Delta_{\max}}\right)$

$\underset{\sim}{x}_i$ spatial coordinate of the ith point in the field

X random field

$X_i = X(\underset{\sim}{x}_i)$ the random field value at location $\underset{\sim}{x}_i$

δ_i foundation settlement as ratio of load to spring constant $\left(\delta_i = F_i / K_i\right)$

$\Delta_{\max}$ maximum differential settlement in a pile group

θ common isotropic correlation length of the random spring constant and load fields

θ' scaled common isotropic correlation length of the random spring constant and load fields ($=\theta / s$)

$\theta_{\ln K}$ isotropic correlation length of the random spring constant field

$\theta_{\ln F}$ isotropic correlation length of the random load field

μ_K mean spring constant

μ_F mean load

μ_X mean of the random variable X

$\mu_{\Delta_{\max}}$ mean maximum differential settlement

$\mu_{\ln K}$ mean of the random variable $\ln K$

$\mu_{\ln F}$ mean of the random variable $\ln F$

$\mu_{\ln X}$ mean of the random variable $\ln X$

$\mu_{\ln \Delta_{\max}}$ mean maximum ln-differential settlement

$\rho_{ij} = \rho\left(\tau_{ij}\right)$ correlation coefficient between X at two points

σ_K standard deviation of spring constant

σ_F standard deviation of load

σ_X standard deviation of the random variable X

$\sigma_{\Delta_{\max}}$ standard deviation of maximum differential settlement

$\sigma_{\ln K}$ standard deviation of ln-spring constant

$\sigma_{\ln F}$ standard deviation of the ln-pile load

$\sigma_{\ln X}$ standard deviation of the random variable $\ln X$

$\sigma_{\ln \Delta_{\max}}$ standard deviation of the maximum ln-differential settlement

τ_{ij} lag distance between i^{th} and j^{th} points in the field

$\Phi(.)$ standard normal cumulative distribution function