Kinematic element method for slope stability

Dieter Stolle & Peijun Guo
McMaster University, Hamilton, Ontario, Canada
Adnan Kader
Wood Canada Ltd, Oakville, Ontario, Canada.

ABSTRACT
Various approaches have been developed for analysing slope stability, including the well-known Fellenius and Bishop's simplified procedures, as well as the more rigorous procedures due to Spencer and Morgenstern & Price. Gussmann (1982) introduced the kinematic element method (KEM) that is based on non-associated plasticity that easily accommodates non-circular failure surfaces. It rigorously accounts for the kinematics and statics of failure to provide an upper bound solution. This paper presents a simplified version of the KEM and proposes an extension to visco-plasticity that is intended to allow the accommodation of force redistribution due to creep. Examples are given to compare the KEM solutions to those provided by the traditional slope stability strategies.

RÉSUMÉ
Diverses approches ont été développées pour analyser la stabilité des pentes, y compris les procédures simplifiées bien connues de Fellenius et Bishop, ainsi que les procédures plus rigoureuses dues à Spencer et Morgenstern & Price. Gussmann (1982) a introduit la méthode des éléments cinématiques (KEM) basée sur une plasticité non associée qui s'adapte facilement aux surfaces de rupture non circulaires. Cette méthode tient rigoureusement compte de la cinématique et de la statique de l'échec à fournir une solution de limite supérieure. Cet article présente une version simplifiée du KEM et propose une extension à la visco-plasticité qui est destinée à permettre l'accommodation de la redistribution des forces dues au fluage. Des exemples sont donnés pour comparer les solutions KEM à celles fournies par les stratégies traditionnelles de stabilité des pentes.

1 INTRODUCTION

An important class of problems in geotechnology is slope stability, which is most often analyzed using the method of slices within a limit equilibrium method (LEM) framework. Various analysis procedures have been developed including, for example, those of Fellenius (1926), Janbu (1954), Bishop (1955), and Morgenstern and Price (1965). The main difference between them is how they deal with the interslice forces and moments. According to Pyke (2017), methods that do "fully satisfy equilibrium" such as those of Spencer (1967) tend to be preferred by both academics and practitioners. Nevertheless, he demonstrated that ordinary method of slices, which only considers global moment equilibrium, does not suffer spurious predictions that are associated with the location of the line of thrust.

Although we can deal with complex stratigraphy, pore pressure distributions and mechanisms the variation in properties can still present problems. Krahn (2003) in his 2001 R.M. Hardy address went as far as to point out that the method of slices, developed for the situation where the normal (and shear) stresses along the slip surface is primarily influenced by gravity, has been pushed far beyond the initial intended purpose. He further suggests that the absence of strain and stress compatibility handicaps the limit equilibrium method. To partially mitigate this handicap, Stolle and Guo (2008) developed a LEM that allows for velocity-dependent sliding between rigid slices.

Models based on the method of slices tend to be oversimplified at times thereby missing important physics, which can lead to questionable predictions. If details are important then finite element analysis, which lends itself well to the analysis of non-homogeneous bodies and design details such as the presence of anchors, is a much better tool; see, e.g. Griffiths and Lane (1999). An excellent overview of slope stability analysis is presented in the book by Duncan and Wright (2005).

From a design point of view, we should keep the process as simple as possible but no less, as we may not properly capture the salient features of the problem. The challenge is achieving a good balance. The method of slices is satisfactory for some problems, such as those where details are not important. A powerful procedure not receiving much attention is the kinematic element method

(KEM) developed by Gussmann (1982). The objective of this paper is to present his procedure and make comparisons with the method of slices.

2 BACKGROUND

We borrow heavily from the publications of Gussmann (1982, 1996, 2000), but change the notation to one that is more in line with that commonly adopted in the method of slices. For this exposition, we keep the presentation simple, by considering only gravity loading and no tension cracks emanating from the surface, as well as neglecting the influence of water. The body is assumed to the homogeneous and of unit width.

Gussmann's method is based on an upper bound collapse theorem presented by Gudehus (1972), which states "A body cannot fail if no kinematically admissible virtual displacement field (satisfying the kinematic boundary conditions and compatible with the flow rule of the material) exists for which an excess kinetic energy is produced." The advantage of this plasticity collapse theorem is that it can be applied with a non-associative flow rule as long as one proceeds kinematically and statically correct. Slip circle analysis is a special case of an upper bound solution (if properly formulated). The previously mentioned factors that are neglected for purposes of clarification can be included; however, there is no guarantee that the predicted factor of safety is an upper bound.

2.1 Kinematic Element Method

There are two main tasks involved in the limit equilibrium method: (a) selecting a failure mechanism and using statics and determining the factor of safety (F_s) via static analysis; and (b) repeating the first task for various potential mechanisms to identify the minimum F_s. An important aspect of these tasks is subdividing the domain of interest and then allocating the material properties to each slice (or element).

Regarding Task (a), which is the focus of this paper, various analysis methods have been developed with the main difference lying in the assumptions associated with the interslice forces. Task (b) may be considered an optimization problem subject to constraints where the objective function is to minimize F_s, subject to constraints. Thus, it is advantageous to use an intelligent (efficient) search routine of which there are many. For the research carried out by the authors, the genetic algorithm implemented by Karchewski et al. (2011) was adopted. Gussmann uses Particle Swarm Optimization; see, e.g., Cheng et al. (2007).

2.1.1 Kinematics

Borrowing from Gussmann's approach, we require a well-defined mechanism to establish the direction of shear at the interface between elements. Shear develops at the interface between elements, except for a circular slip of a homogeneous body, which allows for rigid-body rotation. The interface forces have a direct influence on the distribution of normal and shear forces along the base, known as the failure (or critical) surface. In KEM, only translation modes are allowed. Without rotation there is no energy associated with the moments and therefore moments need not be considered. It is assumed that the locations of the application of forces implicitly take care of themselves. This may restrict movement but simplifies the analysis

Similar to limit equilibrium analyses a potential failure surface must be assumed, a priori. Let us begin with the slope problem shown in Figure 1(a), where we have only one "row" of four elements, supported by a perfectly rigid material. A simple discretization allows us to keep an eye on the assumptions and to simplify the resulting equations. The introduction of more rows is beyond the scope of this paper. Kinematic elements are rigid with deformation (relative sliding) only taking place along the boundary between the elements and along the slip surface 1-3-5-7-9. We assume that there is no dilation at the interfaces, which implies that the normal relative velocities between neighboring elements and perpendicular to the slip surface are zero.

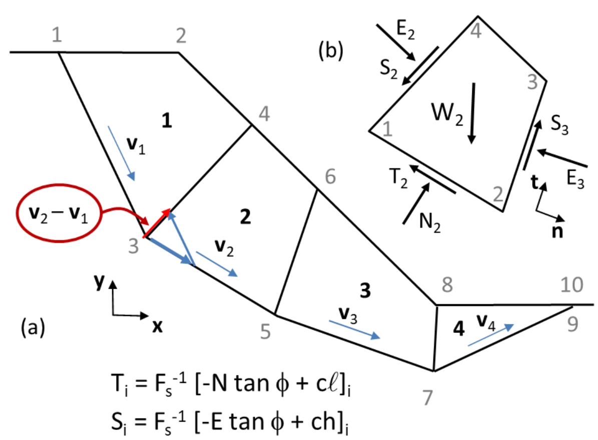

Figure 1: Definition of slope problem for simplified mesh: (a) Geometry and element numbering; and (b) Statics

Let us look what happens between elements1 and 2. The element velocities along edges 1-3 and 3-5 are $\mathbf{v}_1 = v_1(c_1, s_1)$ and $\mathbf{v}_2 = v_2(c_2, s_2)$, respectively, where $(c_i, s_i) = (\cos\alpha_i, \sin\alpha_i)$, with the index i denoting the corresponding element number and α_i being the angle of the velocity vector relative to the x-axis. If we let the unit vector for interface 3-4 be defined as $\mathbf{t}_t = (c_t, s_t) = (\cos\alpha_t, \sin\alpha_t)$, then we can determine v_2 in terms of v_1 and the velocity of element 2 relative 1 (Δv_{21}) in the direction $\mathbf{t}_t$ as

$$v_2 = v_1 \frac{\sin(\alpha_t - \alpha_1)}{\sin(\alpha_t - \alpha_2)}; \text{ etc.} \qquad [1]$$

and

$$\Delta v_{21} = v_1 \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\alpha_t - \alpha_2)}; \;\; \text{etc.} \qquad [2]$$

Figure 1(a) demonstrates $\Delta \mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1$ graphically. An examination of these equations indicates that if we define $v_1 = 1$, then we know the velocities of all elements, as well as the relative velocities. The definition of the mechanism is important with regards to defining the orientation of the shear forces. The shear directions shown in Figure 1 correspond to the slope moving from left to right.

2.1.2 Statics

Once the mechanism has been identified and direction of shear on each interface is established, the next task is to identify the factor of safety F_s that reduces the limiting shear forces to a point where the equilibrium can be satisfied where the Coulomb failure condition is assumed to apply; i.e.,

$$|T| = \frac{N \tan\phi + C}{F_s} \qquad [3]$$

in which T represents the mobilized shear resistance, N is the normal force on the slip surface, $C = c\ell$, with ϕ and c denoting the friction angle and cohesion, respectively, ℓ representing the length of a slice at its base and F_s being the factor of safety. This criterion is also assumed to apply to each interface, with *T* and *N* being replaced by *S* and *E*, respectively. No distinction is made here between total and effective stress, although strictly speaking failure depends on the effective stresses.

Referring to Figure 1(b) and assuming that there are no surface forces, equilibrium yields

$$\begin{aligned} -N_2\mathbf{n}_{1-2} - T_2\mathbf{t}_{1-2} - E_3\mathbf{n}_{2-3} + S_3\mathbf{t}_{2-3} - E_2\mathbf{n}_{4-1}\cdots \\ + S_2\mathbf{t}_{4-1} = W_2\,(0,1) \end{aligned} \qquad [4]$$

in which, for example, $\mathbf{n}_{1\text{-}2}$ and $\mathbf{t}_{1\text{-}2}$ denote the outward normal and tangent directions along local boundary 1-2, respectively, assuming the right-hand-rule convention. The term associated with brackets on the right-hand side is a vector expressed as an order pair. Applying similar operations to all elements leads to a system of linear equilibrium equations in terms of forces that can be reduced to a matrix form, where the normal forces and F_s remain as the unknowns given the Coulomb constraints and the known boundary conditions along 1-2 and 9-10. A useful relation between normal **n** and tangential **t** directions is $(n_x, n_y) = (t_y, -t_x)$ where the terms in brackets represent the direction cosines relative to the x-y frame of reference.

2.1.3 Search for Global Minimum

The resulting matrix equation is nonlinear in F_S. Thus, an iterative procedure is required to identify the root that delivers the minimum factor of safety for the given mechanism that properly satisfies the physics. The reader is referred to Gussmann (2000), who provides an efficient algorithm for this task. Since the assumed mechanism does not necessarily correspond to the global minimum F_s, similar analyses are repeated assuming other failure patterns. Within the context of KEM, all interfaces are potential slip surfaces. As a result, when searching for the critical failure mechanism, all vertices except those defining the slope are variables in the optimization process. The moveable vertices along the surface must remain on the surface. Furthermore, should the domain of interest be heterogeneous, it is necessary to find average values of the properties along each interface and along the basal slip surface.

2.2 Limit Equilibrium Method

The equilibrium equations that we obtained by introducing the failure equations and direction cosines into Eq. (4) are rather long. In this section we develop the corresponding equation for the method of slices where it is assumed that the lateral shear is given by the Spencer assumption $S = \lambda(E - U)$, in which λ, E and U denote scaling factor, total horizontal force on an interface. and corresponding water force, respectively. The scaling factor is a constant that corresponds approximately to the slope of the surface. Along the base we have $T = [(N - U_b)\tan\phi + c\ell]/F_s$, in which N and U_b are the total normal force and basal water force, respectively. The other terms are the same as defined previously

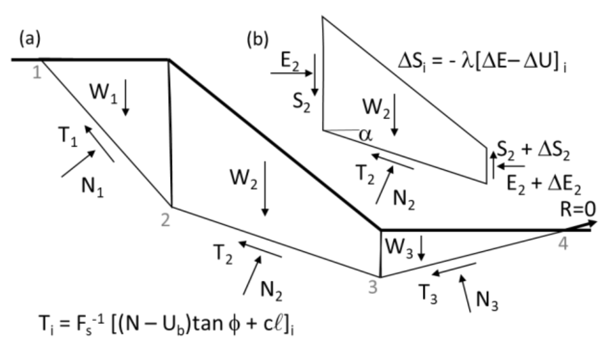

Figure 2: Definitions for method of slices: (a) Geometry and element numbering; and (b) Statics

Referring to Figures 2(a) and (b), we have a slope subdivided into 3 slices and free body diagram for slice 2. The notation is simplified and since both lateral slices are vertical, we can write equilibrium for a single slice in incremental form as,

$$\begin{bmatrix} \sin\alpha + t_m \cos\alpha & 1 \\ \cos\alpha - t_m \sin\alpha & \lambda \end{bmatrix}_I \begin{bmatrix} N \\ \Delta E \end{bmatrix}_I = \cdots$$

$$\begin{bmatrix} \cos\alpha\left(t_m U_b - C_m\right) \\ W + \lambda\Delta U - \sin\alpha\left(t_m U_b - C_m\right) \end{bmatrix}_I \quad [5]$$

in which Δ denotes an increment, $t_m = \tan\phi/F_s$ and $C_m = c\ell/F_s$, with ℓ being the length of the base of a slice. Given that the height of one edge of a slice is zero at the beginning and at the end of the slip surface, and $\sum_i \Delta E_i = 0$, we can write after algebraic manipulation,

$$F_s = \frac{\sum_i \frac{1}{D_i}\left[\cos\alpha \tan\phi\left(W + \lambda\Delta U\right) + c_f\ell - \tan\phi U_b\right]_i}{-\sum_i \frac{1}{D_i}\left[\sin\alpha\left(W + \lambda\Delta U\right)\right]_i}$$

with [6]

$$D_i = \left[\left(\lambda\sin\alpha - \cos\alpha\right) + \left(\lambda\cos\alpha + \sin\alpha\right)\frac{\tan\phi}{F_s}\right]_i \quad .$$

We could just as well have adopted KEM and assumed that the vertical faces satisfy the Coulomb criterion. However, experience has shown that the factor of safety tends to be too high when forcing vertical interfaces, given that it is unlikely that all vertical interfaces would correspond to slip lines.

3. NUMERICAL EXAMPLES

Referring to Figure 3, we consider an example addressed in the classic paper by Fredlund & Krahn (1977), in which three cases were considered: (a) 12 m high, 2-1 slope resting on 7.5 m thick subgrade of similar uniform material (Soil 1); (b) Case (a) with the addition of a 1 m thick weak layer (Soil 2), 1.5 m below the subgrade; and (c) Case (a) with a 1-10 sloping piezometric surface that meets the subgrade surface at the toe of the slope. The saturated unit weight is assumed to be the same as the dry unit weight. Table 1 provides the soil properties.

Figure 3 shows the critical surfaces for each case computed by the limit equilibrium method (Karchewski, 2012) along with the critical mechanisms predicted by the kinematic element method (Gussmann 2000). The factors of safety are summarized in Table 2, along with published results by (Fredlund and Krahn, 1977).

Table 1. Slope properties (Fredlund and Krahn, 1977)

Material	ϕ (º)	c (kPa)	γ (kN/m^3)
Soil1	20	28.7	18.85
Soil 2	10	0	18.85

Referring to Table 2, we observe that the predicted factors of safety are slightly less than those published for all cases. The predictions by LEM correspond to those in which slices were used. The results with 36 slices were virtually the same given that the solutions for F_s are not unique when using a genetic algorithm. The parameter λ is a system (or fitting) parameter and not a material property. For the three cases shown in Figure 3. The respective values were approximately 0.31, 0.15 and 0.33. All simulations assumed that S = λ E (Spencer approximation).

The published results are also based on the Morgenstern-Price method. Except for the case of the slope with a weak layer the solutions correspond to circular failure surfaces. We clearly observe when examining Figure 3(a) that the classical assumption of circular failure is reasonable for a soil with uniform and relatively isotropic properties. The objective of analyzing Case (b) was to determine whether the KEM is capable of identifying the block sliding mechanism, which is assumed to be the correct failure mechanism for this slope. The KEM does in fact locate the block sliding mechanism and provides a critical failure surface similar to that predicted LEM.

As might be expected, the addition of a water table for Case (c) reduces the effective stress in the soil, thereby reducing the factor of safety by approximately 37%. The failure mechanism is altered from a toe failure to a deeper-seated failure, with the critical slip surface obtained using the KEM being deeper. The addition of more elements did not shift the failure surface upwards.

Table 2. Comparison of factors of safety, F_s

Case	LEM	KEM	Spencer	M&P
(a)	1.98	1.96	2.07	2.08
(b)	1.24	1.24	1.37	1.38
(c)	1.78	1.81	1.83	1.83

The question arises, what is so special about the kinematic element method? As indicated previously, the procedure is based on a collapse theorem that provides an upper bound. Unlike the limit equilibrium approach, in which the relation for interslice shear is chosen for algebraic convenience, the lateral interfaces for KEM represent slip lines that depend on the Coulomb failure criterion.

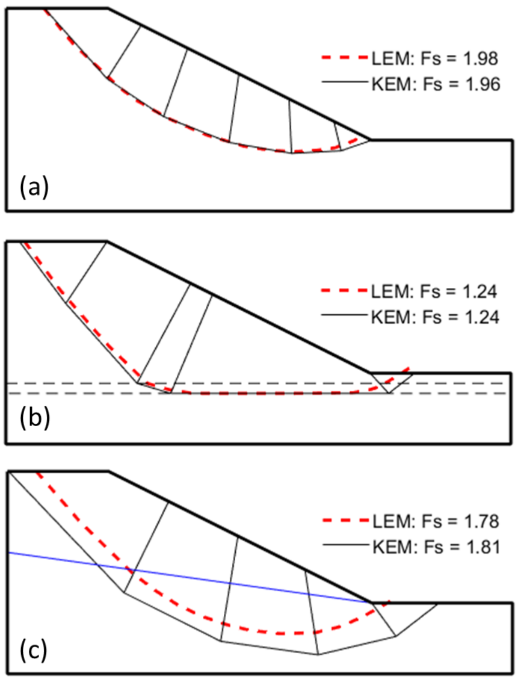

Figure 3: Critical slip surfaces for Cases (a), (b) and (c).

A second advantage is the smaller number of elements that are required, which makes the failure mechanism more transparent. Figure 4 shows results corresponding to a simple mechanism for Case (a), which has an F_s = 2.01. The forces are well defined, although where they act along the element boundary remain unknown. Nevertheless, all the information that we require is easily available, without additional assumptions being necessary.

For KEM, the motion is restricted to translational, which is why it is not necessary to consider moments as indicated earlier. Each slice in the LEM represents a narrow body. We do not take into account how the forces are transmitted along the slice. Based on experience, we observed that the predictions of basal forces in the LEM are sensitive to small changes in the interslice force distribution That is, a small variations in the parameter λ can lead to a significant change in the normal force and shear force distributions along the base.

4. INFLUENCE OF DEFORMATION

The traditional limit equilibrium method has advantages in its simplicity. On the other hand, a very important aspect that is lost is the influence of creep on load redistribution within the slope and on the corresponding factor of safety. To accomplish this, it is necessary to take into account a coupling between the kinematics and equilibrium. An attempt to partially accomplish this was presented by Stolle and Guo (2008), in which they used nonlinear sliding relations to take into account "deformation" influences. In this section we propose an alternative approach to that presented previously.

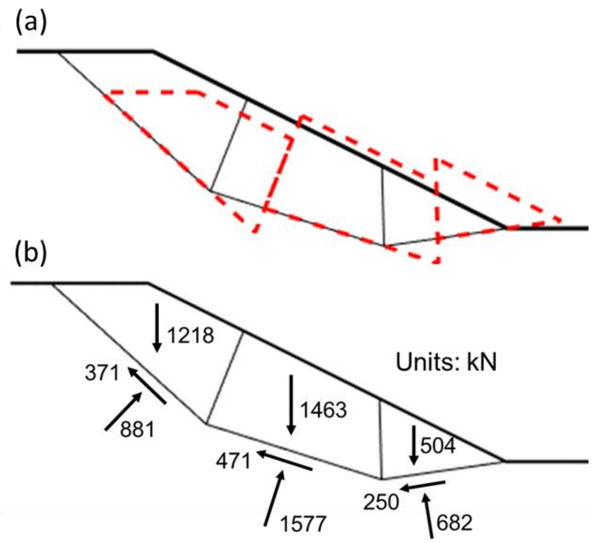

Figure 4:(a) Mechanism and (b) forces for Case (a)

Borrowing from Zienkiewicz and Godbole (1974), Stolle et al. (2004) developed a visco-plasticity approach to model the potential for slope instability. Using the same logic, the following constitutive law can be derived,

$$T = \bar{\mu}\Delta v \ell + \frac{(N\tan\phi + c\ell)}{F_s}\frac{\Delta v}{|\Delta v|} \qquad [7]$$

in which $\bar{\mu}$ is a pseudo-viscosity term that represents the minimum possible value for viscosity and Δv is the relative velocity between the two sides of an interface similar to what we have in Eq. (1) and Eq. (2). The other terms are the same as defined previously. If an appropriate value for $\bar{\mu}$ is selected, we can proceed as we did previously to determine F_s with the KEM equivalent to Eq. (5) since $\bar{\mu}\Delta v$ is known and is moved to the right-hand side. The kinematics automatically account for the direction of T through the term $\Delta v/|\Delta v|$. Referring to Eq. (6), the effect of the deformation term on F_s is obtained by replacing the term $[\sin\alpha(W + \lambda\Delta U)]$ with $[\sin\alpha(W + \lambda\Delta U) + \bar{\mu}\ell v]$. Although it is not obvious, $\sin\alpha$ negative along much of the basal boundary, whereas $\bar{\mu}\ell v$ is positive. This has the net affect of increasing the factor of safely.

We must remember that the objective of the exercise is to find F_s that allows the given mechanism to form.

5 CONCLUDING REMARKS

As was indicated at the beginning of the paper, the primary objective was to present a simplified kinematic element method for slope stability. An important aspect of the procedure is that no assumptions must be made, a priori, regarding the treatment of the inter-element forces and that an important part of the solution procedure is to identify the orientation of the lateral surfaces to ensure that they correspond to "slip lines". The quotation marks are intended to emphasize that the lines correspond to factored strength parameters.

It should be mentioned that KEM requires an understanding of mechanics and should not be used as a black box.

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